Multiaxial Fatigue

Introduction

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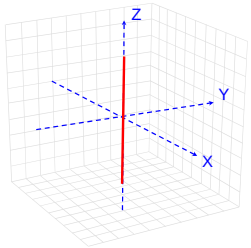
When is Multiaxial Fatigue Important ?

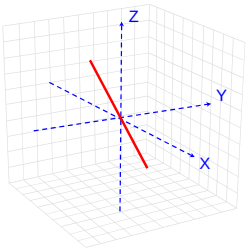
- Complex state of stress
- Complex out of phase loading

Uniaxial Stress



one principal stress
one direction

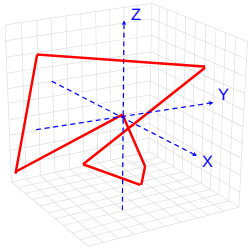
Proportional Biaxial



principal stresses vary
proportionally
but do not rotate

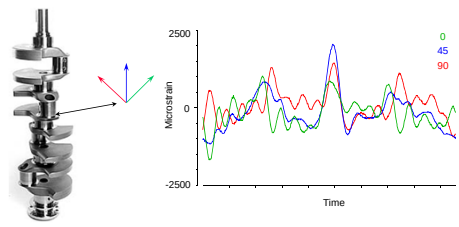
$\sigma_1 = \alpha\sigma_2 = \beta\sigma_3$

Nonproportional Multiaxial

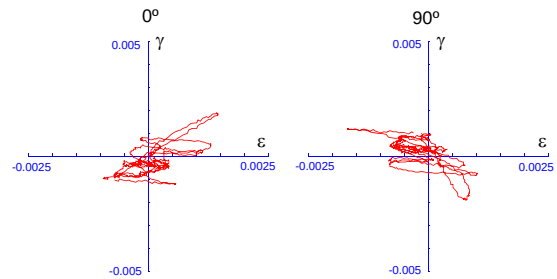


Principal stresses may
vary nonproportionally
and/or change direction

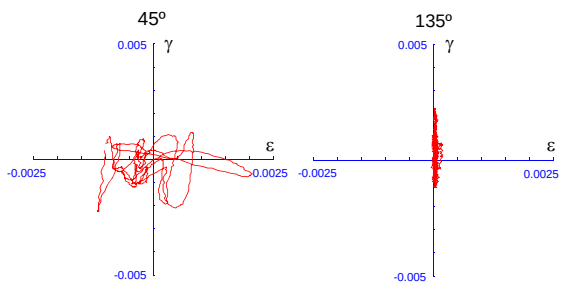
Crankshaft



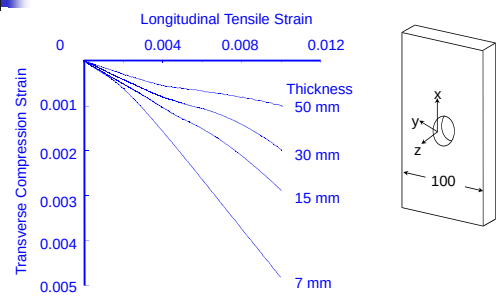
Shear and Normal Strains



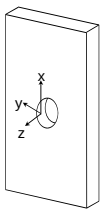
Shear and Normal Strains



3D stresses



Notch Stresses

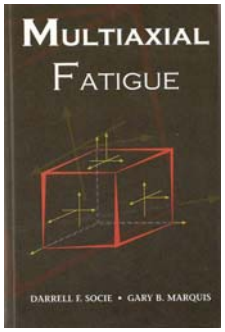


t	ε_x	ε_z	σ_x	σ_z
7	0.01	-0.005	63.5	0
15	0.01	-0.003	70.6	14.1
30	0.01	-0.002	73.0	21.8
50	0.01	-0.001	75.1	29.3

Outline

- State of Stress
- Stress-Strain Relationships
- Fatigue Mechanisms
- Multiaxial Testing
- Stress Based Models
- Strain Based Models
- Fracture Mechanics Models
- Nonproportional Loading
- Stress Concentrations

Book



Outline

- **State of Stress**
- Stress-Strain Relationships
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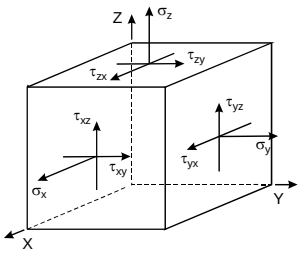
State of Stress

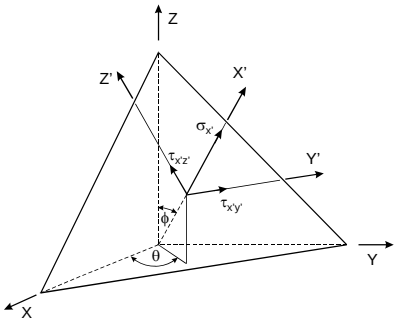
- Stress components
- Common states of stress
- Shear stresses

Stress Components

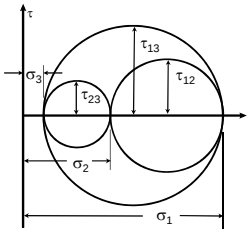


Six stresses and six strains

Stresses Acting on a Plane

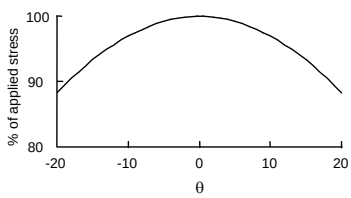


Principal Stresses



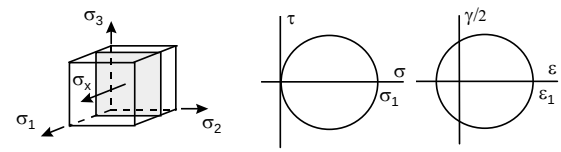
$$\sigma^3 - \sigma^2(\sigma_X + \sigma_Y + \sigma_Z) + \sigma(\sigma_X\sigma_Y + \sigma_Y\sigma_Z\sigma_X\sigma_Z - \tau^2_{XY} - \tau^2_{YZ} - \tau^2_{XZ})$$
$$- (\sigma_X\sigma_Y\sigma_Z + 2\tau_{XY}\tau_{YZ}\tau_{XZ} - \sigma_X\tau^2_{YZ} - \sigma_Y\tau^2_{ZX} - \sigma_Z\tau^2_{XY}) = 0$$

Stress and Strain Distributions



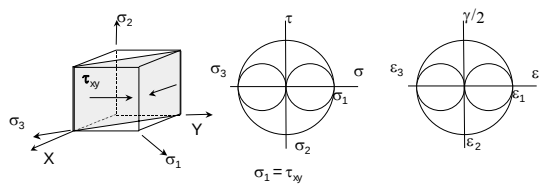
Stresses are nearly the same over a 10° range of angles

Tension



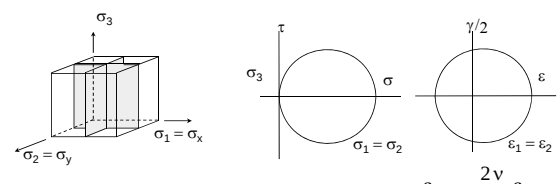
$\sigma_2 = \sigma_3 = 0$ $\varepsilon_2 = \varepsilon_3 = -\nu\varepsilon_1$

Torsion



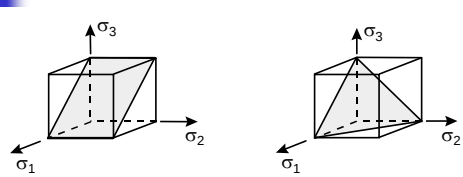
$\sigma_1 = \tau_{xy}$

Biaxial Tension



$$\varepsilon_3 = -\frac{2\nu}{1-\nu}\varepsilon$$

Shear Stresses

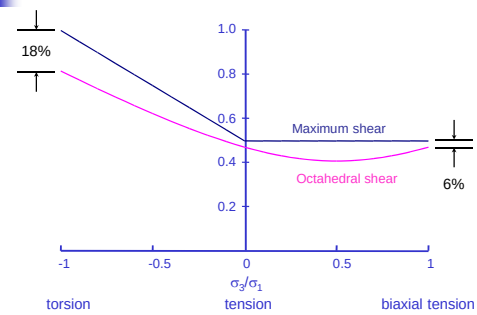


Maximum shear stress Octahedral shear stress

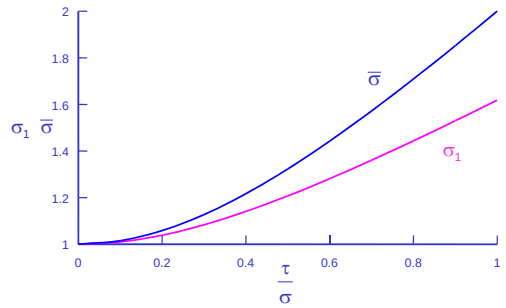
$$\tau_{13} = \frac{|\sigma_1 - \sigma_3|}{2} \qquad \tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1-\sigma_3)^2+(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2}$$

Mises: $\bar{\sigma} = \frac{3}{\sqrt{2}}\tau_{oct}$ $\tau_{oct} = \frac{3}{2\sqrt{2}}\tau_{13} = 0.94\,\tau_{13}$

Maximum and Octahedral Shear



Shear Stress Influence



State of Stress Summary

- Stresses acting on a plane
- Principal stress
- Maximum shear stress
- Octahedral shear stress

Outline

- State of Stress
- **Stress-Strain Relationships**
- Fatigue Mechanisms
- Multiaxial Testing
- Stress Based Models
- Strain Based Models
- Fracture Mechanics Models
- Nonproportional Loading
- Stress Concentrations

Jenkin 1922

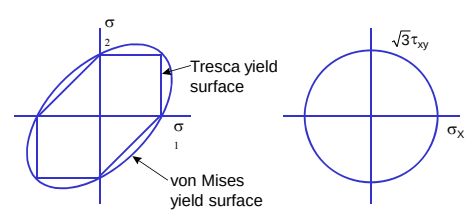
"About six months ago I wrote a paper, knowing that I should be very busy in the autumn and made a model to illustrate a point in it. But as I played with the model to learn how to use it, it grew too strong for me and took command and for the last six months I have been its obedient slave --- for the model explained the whole of my subject *Fatigue*."

"Fatigue in Metals," *The Engineer*, Dec. 8, 1922

Elastic Stress Strain Relationships

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix}$$

Yield Surfaces



Equations

Tresca

$$F_1 = \sigma_1 - \sigma_3 - \sigma_{ys} = 0$$

$$F_2 = \sigma_1 - \sigma_2 - \sigma_{ys} = 0$$

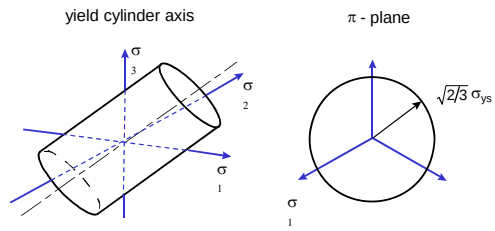
or

$$F_3 = \sigma_2 - \sigma_3 - \sigma_{ys} = 0$$

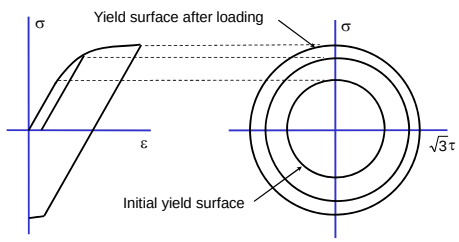
Mises

$$F = \sigma_x^{\,2} + \sigma_y^{\,2} - \sigma_x\,\sigma_y + 3\tau_{xy}^{\,2} - \sigma_{ys}^{\,2} = 0$$

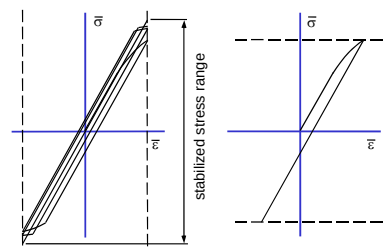
Mises Yield Surfaces



Isotropic Hardening

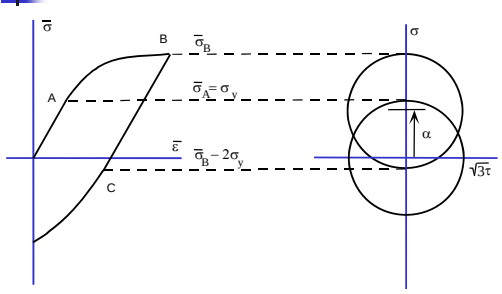


Cyclic Loading

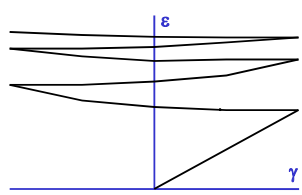


Strain Control Stress Control

Kinematic Hardening

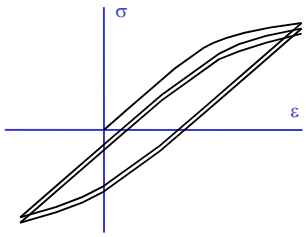


Ratcheting



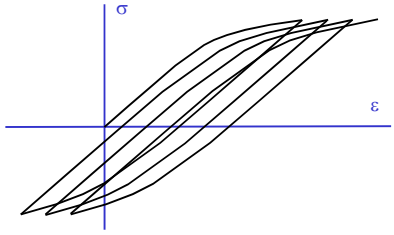
Cyclic torsion with a mean tension stress

Cyclic Mean Stress Relaxation



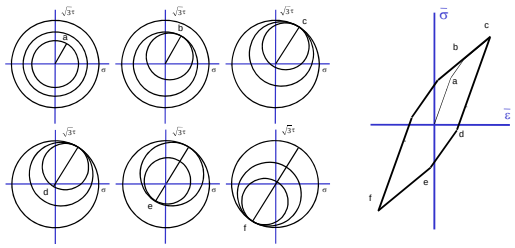
Cyclic strain with mean stress

Cyclic Creep

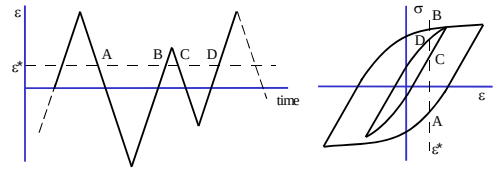


Cyclic stress with a mean stress

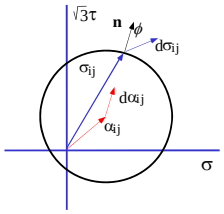
Plasticity Models



Why is modeling needed?



Flow and Hardening Rules



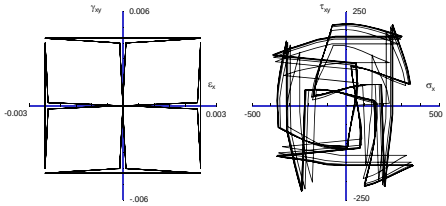
Flow rule

$$d\varepsilon_{ij}^{p} = d\lambda \frac{\partial F}{\partial \sigma_{ij}}$$

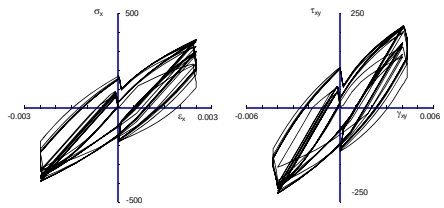
Hardening rule

$$F = f(\sigma_{ij}) + g(\bar{\varepsilon}^{p}) = 0$$

Multiaxial Example



Multiaxial Example



Summary

- Isotropic Hardening
- Kinematic Hardening
- Cyclic creep or ratcheting
- Mean stress relaxation
- Nonproportional hardening

Outline

- State of Stress
- Stress-Strain Relationships
- **Fatigue Mechanisms**
- Multiaxial Testing
- Stress Based Models
- Strain Based Models
- Fracture Mechanics Models
- Nonproportional Loading
- Stress Concentrations

Size Scale for Studying Fatigue

Atoms | Dislocations | Crystals | Specimens | Structures

10^{-10} 10^{-8} 10^{-6} 10^{-4} 10^{-2} 10^{0} 10^{2}

Understand the physics on this scale

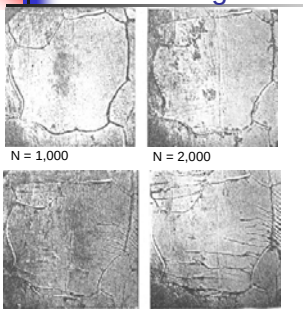
Model the physics on this scale

Use the models on this scale

The Fatigue Process

- Crack nucleation
- Small crack growth in an elastic-plastic stress field
- Macroscopic crack growth in a nominally elastic stress field
- Final fracture

1903 - Ewing and Humfrey



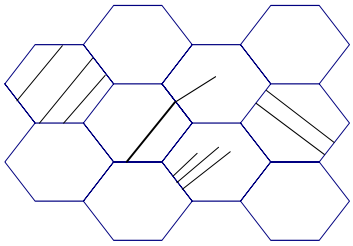
N = 1,000 N = 2,000

N = 10,000 N = 40,000 N_f = 170,000

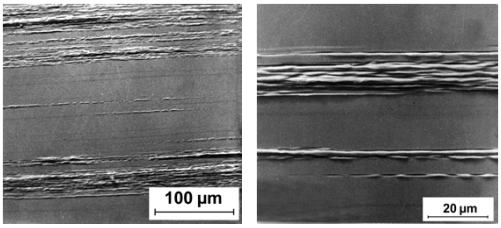
Cyclic deformation leads to the development of slip bands and fatigue cracks

Ewing, J.A. and Humfrey, J.C. "The fracture of metals under repeated alterations of stress", *Philosophical Transactions of the Royal Society*, Vol. A200, 1903, 241-250

Crack Nucleation

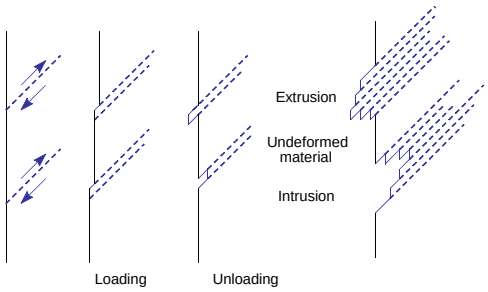


Slip Band in Copper

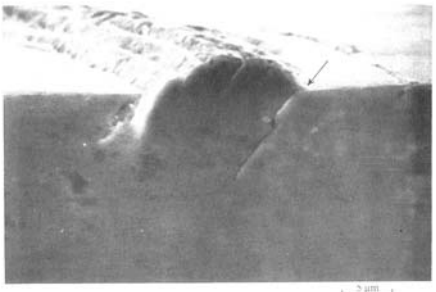


Polak, J. Cyclic Plasticity and Low Cycle Fatigue Life of Metals, Elsevier, 1991

Slip Band Formation

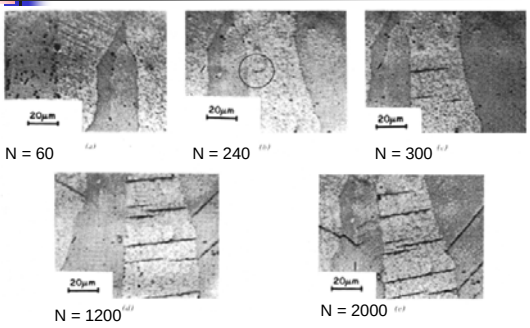


Slip Bands



Ma, B-T and Laird C. "Overview of fatigue behavior in copper sinle crystals –II Population, size, distribution and growth Kinetics of stage I cracks for tests at constant strain amplitude", Acta Metallurgica, Vol 37, 1989, 337-348

2124-T4 Cracking in Slip Bands



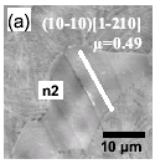
N = 60 N = 240 N = 300

N = 1200 N = 2000

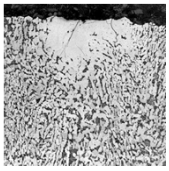
2219-T851 Cracked Particle

James & Morris, *ASTM STP 811 Fatigue Mechanisms: Advances in Quantitative Measurement of Physical Damage*, pp. 46-70, 1983.

Initiation in Ti

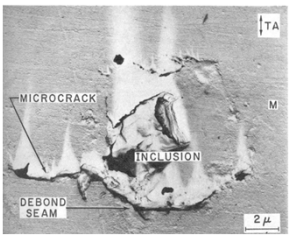


Slip bands



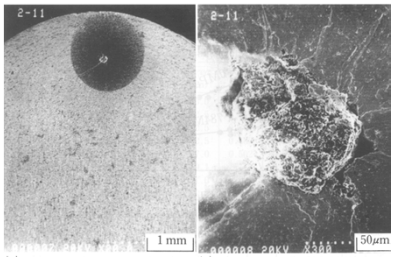
Hard alpha inclusion

Crack Initiation at Inclusions



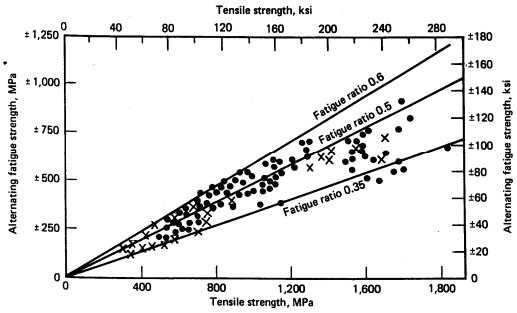
Langford and Kusenberger, "Initiation of Fatigue Cracks in 4340 Steel", *Metallurgical Transactions*, Vol 4, 1977, 553-559

Subsurface Crack Initiation



Y. Murakami, Metal Fatigue: *Effects of Small Defects and Nonmetallic Inclusions*, 2002

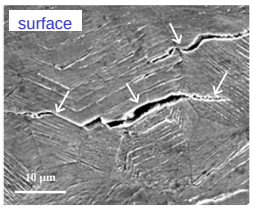
Fatigue Limit and Strength Correlation

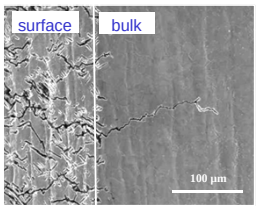


Crack Nucleation Summary

- Highly localized plastic deformation
- Surface phenomena
- Stochastic process

Surface Damage

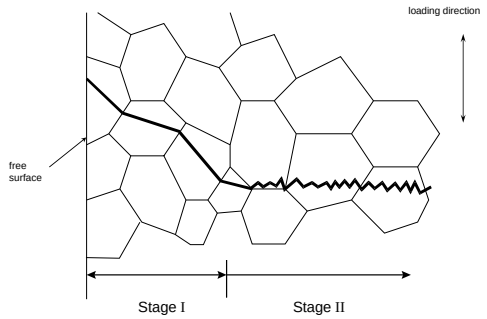




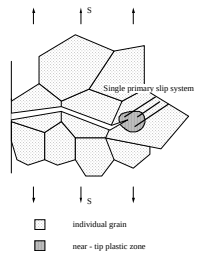
20-25 austenitic steel in symmetrical push-pull fatigue
(20°C, $\Delta\varepsilon_p/2 = \pm 0.4\%$) : short cracks on the surface and in the bulk

From Jacques Stolarz, Ecole Nationale Superieure des Mines
Presented at LCF 5 in Berlin, 2003

Stage I and Stage II

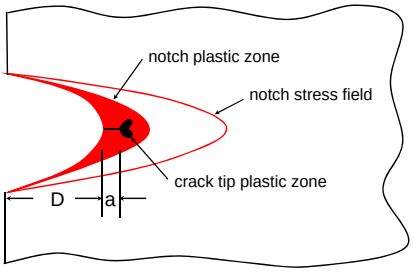


Stage I Crack Growth



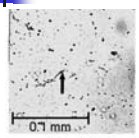
Stage I crack is strongly affected by slip characteristics, microstructure dimensions, stress level, extent of near tip plasticity

Small Cracks at Notches

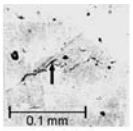


Crack growth controlled by the notch plastic strains

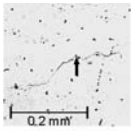
Small Crack Growth



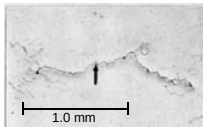
N = 160



N = 240



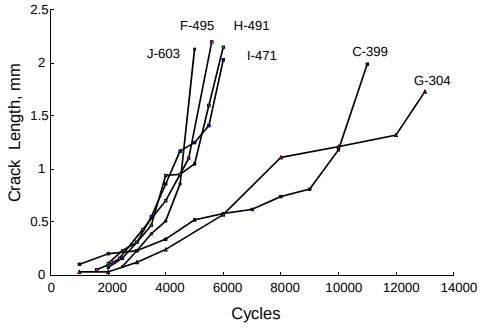
N = 520



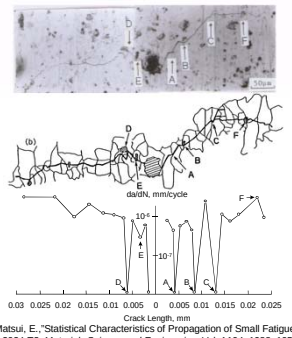
N = 900

Inconel 718
$\Delta\varepsilon = 0.02$
$N_f = 936$

Crack Length Observations

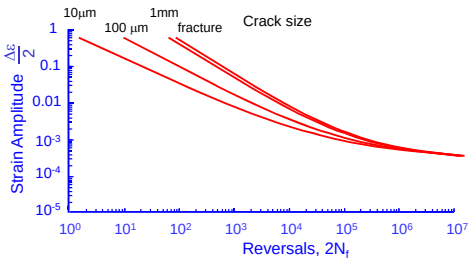


Crack - Microstructure Interactions



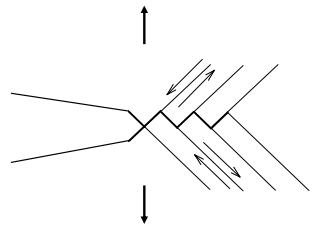
Akiniwa, Y., Tanaka, K., and Matsui, E.,"Statistical Characteristics of Propagation of Small Fatigue Cracks in Smooth Specimens of Aluminum Alloy 2024-T3, *Materials Science and Engineering*, Vol. A104, 1988, 105-115

Strain-Life Data



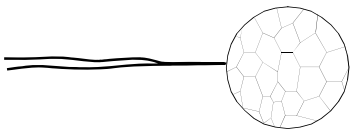
Most of the life is spent in microcrack growth in the plastic strain dominated region

Stage II Crack Growth



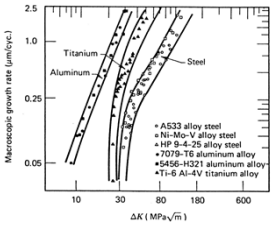
Locally, the crack grows in shear
Macroscopically it grows in tension

Long Crack Growth



Plastic zone size is much larger than the material microstructure so that the microstructure does not play such an important role.

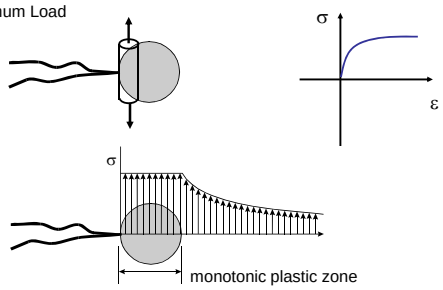
Crack Growth Rates of Metals



Material strength does not play a major role in fatigue crack growth

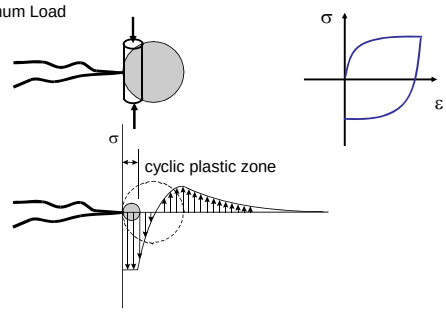
Stresses Around a Crack

Maximum Load

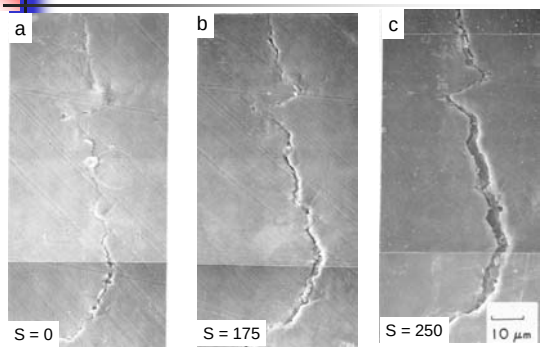


Stresses Around a Crack (continued)

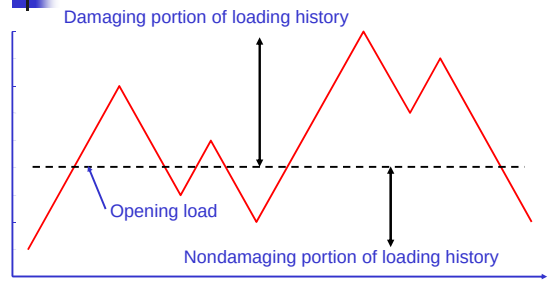
Minimum Load



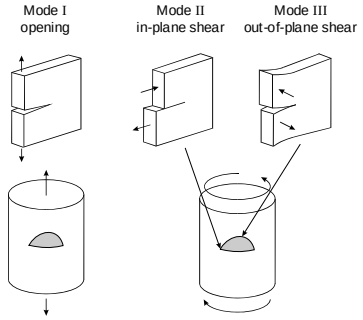
Crack Closure



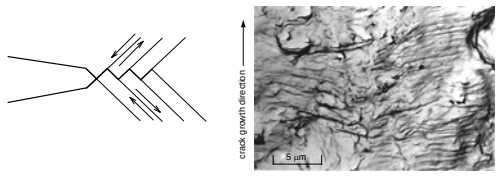
Crack Opening Load



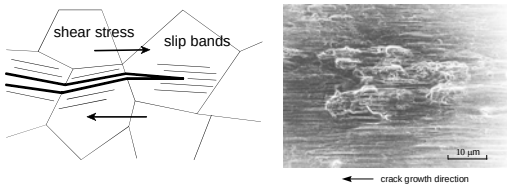
Mode I, Mode II, and Mode III



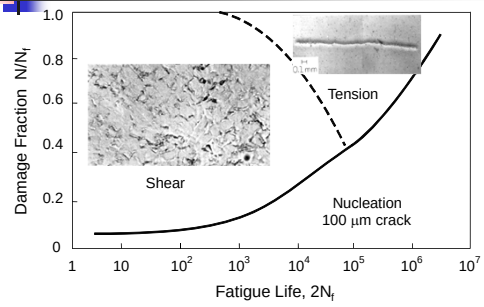
Mode I Growth



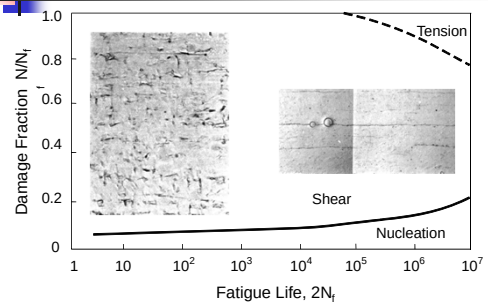
Mode II Growth



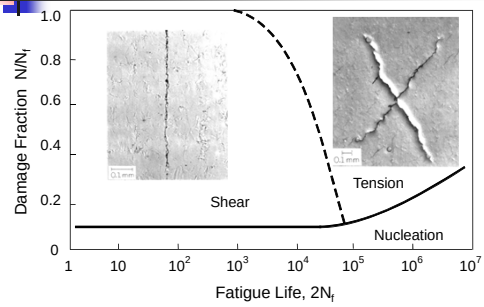
1045 Steel - Tension



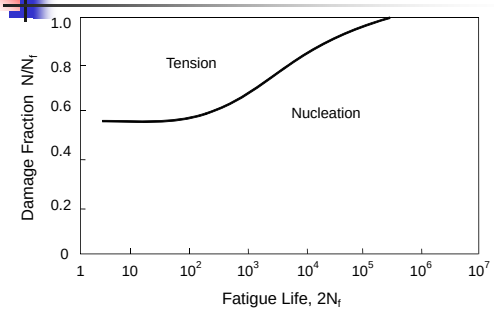
1045 Steel - Torsion



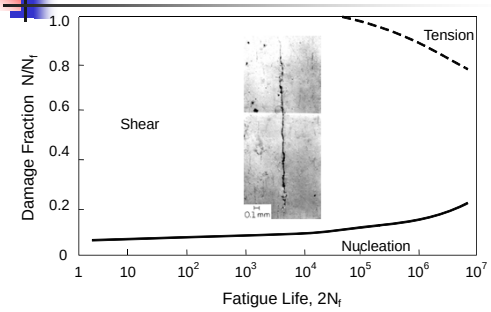
304 Stainless Steel - Torsion



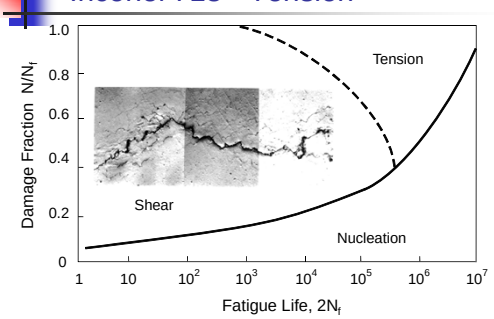
304 Stainless Steel - Tension



Inconel 718 - Torsion



Inconel 718 - Tension

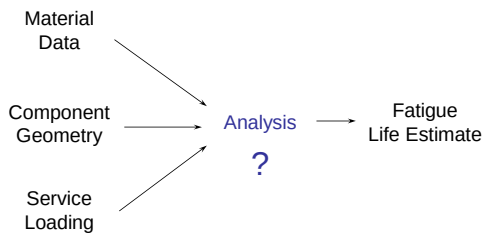


Fatigue Made Easy

Similitude

Professor Darrell F. Socie
Mechanical Science and Engineering
University of Illinois

Fatigue Analysis



The Similitude Concept

Why Fatigue Modeling Works !

What is the Similitude Concept

The "Similitude Concept" allows engineers to relate the behavior of small-scale cyclic material test specimens, defined under carefully controlled conditions, to the likely performance of real structures subjected to variable amplitude fatigue loads under either simulated or actual service conditions.

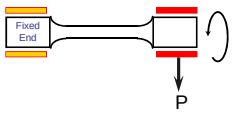
Fatigue Analysis Techniques

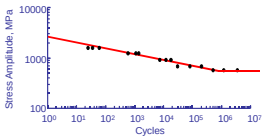
Stress - Life
BS 7608, Eurocode 3
Strain - Life
Crack Growth

Life Estimation

Method	Physics	Size
Stress-Life	Crack Nucleation	0.01 mm
BS 7608	Crack Growth	1 - 10 mm
Strain-Life	Microcrack Growth	0.1 - 1 mm
Crack Growth	Macrocrack Growth	> 1mm

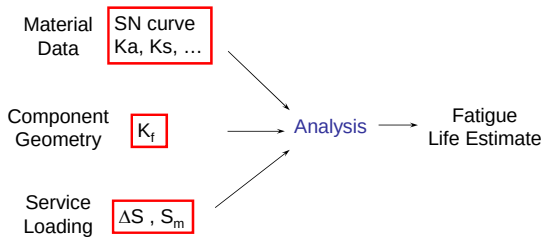
Stress-Life Fatigue Modeling





The Similitude Concept states that if the instantaneous loads applied to the 'test' structure (wing spar, say) and the test specimen are the same, then the response in each case will also be the same and can be described by the material's S-N curve. Due account can also be made for stress concentrations, variable amplitude loading etc.

Fatigue Analysis: Stress-Life



Stress-Life

- Major Assumptions:
  - Most of the life is consumed nucleating cracks
  - Elastic deformation
  - Nominal stresses and material strength control fatigue life
  - Accurate determination of K_f for each geometry and material

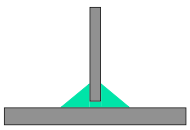
Stress-Life

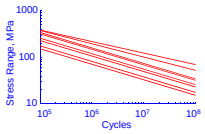
- Advantages:
  - Changes in material and geometry can easily be evaluated
  - Large empirical database for steel with standard notch shapes

Stress-Life

- Limitations:
  - Does not account for notch root plasticity
  - Mean stress effects are often in error
  - Requires empirical K_f for good results

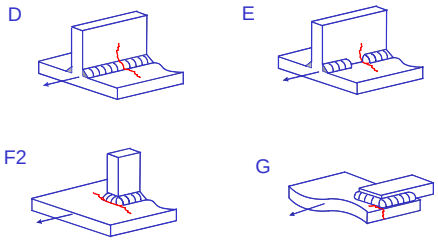
BS 7608 Fatigue Modeling





The Similitude Concept states that if the instantaneous loads applied to the 'test' structure (welded beam on a bulldozer, say) and the test specimen (standard fillet weld) are the same, then the response in each case will also be the same and can be described by one of the standard BS 7608 Weld Classification S-N curves.

Weld Classifications



Fatigue Analysis: BS 7608

Material Data: Weld SN curve

Component Geometry: Class

Service Loading: ΔS

→ Analysis → Fatigue Life Estimate

BS 7608

- Major Assumptions:
  - Crack growth dominates fatigue life
  - Complex weld geometries can be described by a standard classification
  - Results independent of material and mean stress for structural steels

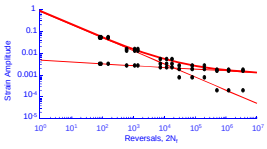
BS 7608

- Advantages:
  - Manufacturing effects are directly included
  - Large empirical database exists

BS 7608

- Limitations:
  - Difficult to determine nominal stress and weld class for complex shapes
  - No benefit for improving manufacturing process

Strain-Life Fatigue Modeling



The Similitude Concept states that if the instantaneous strains applied to the 'test' structure (vehicle suspension, say) and the test specimen are the same, then the response in each case will also be the same and can be described by the material's e-N curve. Due account can also be made for stress concentrations, variable amplitude loading etc.

Fatigue Analysis: Strain-Life

Material Data: εN curve, $\sigma\varepsilon$ curve

Component Geometry: K_f

Service Loading: ΔS , S_m

→ Analysis → Fatigue Life Estimate

Strain-Life

- Major Assumptions:
  - Local stresses and strains control fatigue behavior
  - Plasticity around stress concentrations
  - Accurate determination of K_f

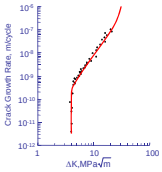
Strain-Life

- Advantages:
  - Plasticity effects
  - Mean stress effects

Strain-Life

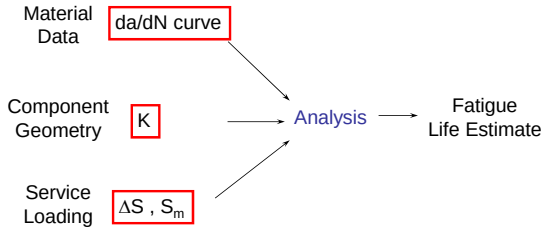
- Limitations:
  - Requires empirical K_f
  - Long life situations where surface finish and processing variables are important

Crack Growth Fatigue Modeling



The Similitude Concept states that if the stress intensity (K) at the tip of a crack in the 'test' structure (welded connection on an oil platform leg, say) and the test specimen are the same, then the crack growth response in each case will also be the same and can be described by the Paris relationship. Account can also be made for local chemical environment, if necessary.

Fatigue Analysis: Crack Growth



Crack Growth

- Major Assumptions:
  - Nominal stress and crack size control fatigue life
  - Accurate determination of initial crack size

Crack Growth

- Advantage:
  - Only method to directly deal with cracks

Crack Growth

- Limitations:
  - Complex sequence effects
  - Accurate determination of initial crack size

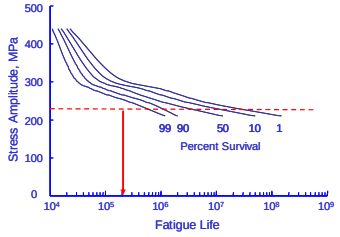
Choose the Right Model

- Similitude
  - Failure mechanism
  - Size scale

Design Philosophy

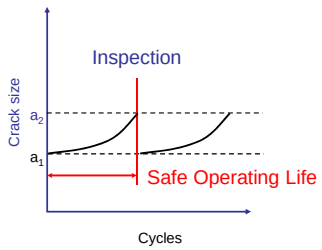
- Safe Life
- Damage Tolerant

Safe Life



Choose an appropriate risk and replace critical parts after some specified interval

Damage Tolerant



Inspect for cracks larger than a_1 and repair

Inspection

A Boeing 777 costs $250,000,000

A new car costs $25,000

For every $1 spent inspecting and maintaining a B 777 you can spend only 0.01¢ on a car

Things Worth Remembering

- Questions to ask
  - Will a crack nucleate ?
  - Will a crack grow ?
  - How fast will it grow ?
- Similitude
  - Failure mechanism
  - Size Scale

Outline

- State of Stress
- Stress-Strain Relationships
- Fatigue Mechanisms
- Multiaxial Testing
- **Stress Based Models**
- Strain Based Models
- Fracture Mechanics Models
- Nonproportional Loading
- Stress Concentrations

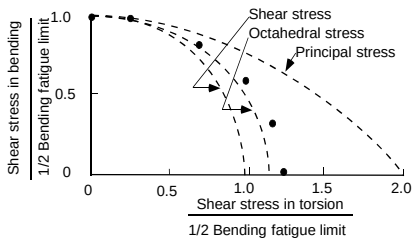
Fatigue Mechanisms Summary

- Fatigue cracks nucleate in shear
- Fatigue cracks grow in either shear or tension depending on material and state of stress

Stress Based Models

- Sines
- Findley
- Dang Van

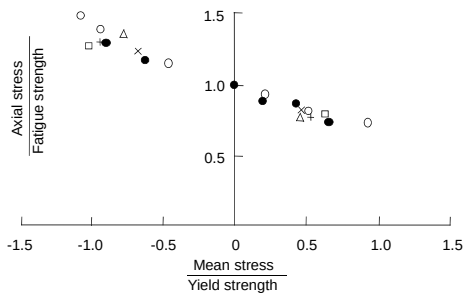
Bending Torsion Correlation



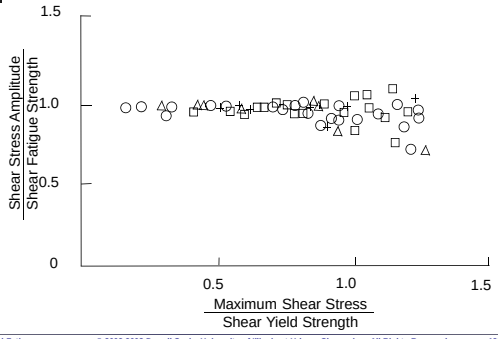
Test Results

- Cyclic tension with static tension
- Cyclic torsion with static torsion
- Cyclic tension with static torsion
- Cyclic torsion with static tension

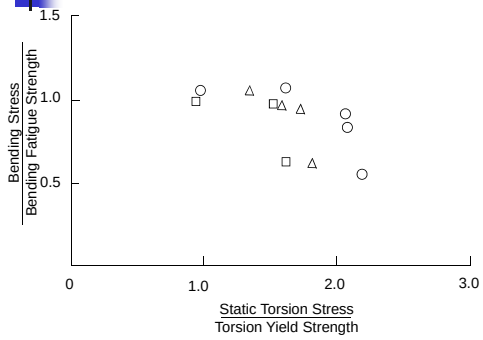
Cyclic Tension with Static Tension



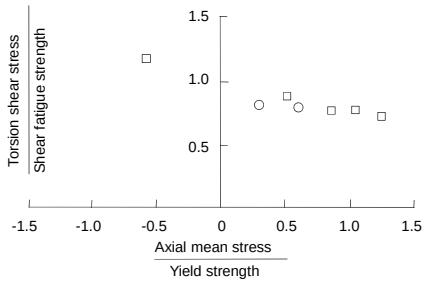
Cyclic Torsion with Static Torsion



Cyclic Tension with Static Torsion



Cyclic Torsion with Static Tension



Conclusions

- Tension mean stress affects both tension and torsion
- Torsion mean stress does not affect tension or torsion

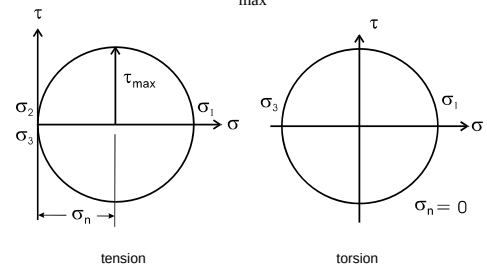
Sines

$$\frac{\Delta\tau_{oct}}{2} + \alpha(3\sigma_h) = \beta$$

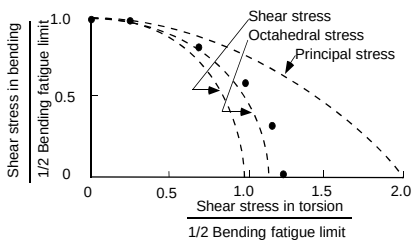
$$\frac{1}{6}\sqrt{(\Delta\sigma_x - \Delta\sigma_y)^2 + (\Delta\sigma_x - \Delta\sigma_z)^2 + (\Delta\sigma_y - \Delta\sigma_z)^2 + 6(\Delta\tau_{xy}^2 + \Delta\tau_{xz}^2 + \Delta\tau_{yz}^2)} + \alpha(\sigma_x^{mean} + \sigma_y^{mean} + \sigma_z^{mean}) = \beta$$

Findley

$$\left(\frac{\Delta\tau}{2} + k\sigma_n\right)_{max} = f$$



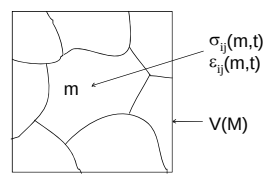
Bending Torsion Correlation



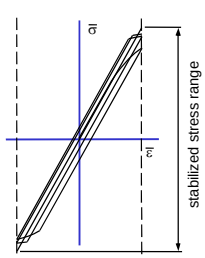
Dang Van

$$\tau(t) + a\sigma_h(t) = b$$

$\Sigma_{ij}(M,t)$ $E_{ij}(M,t)$

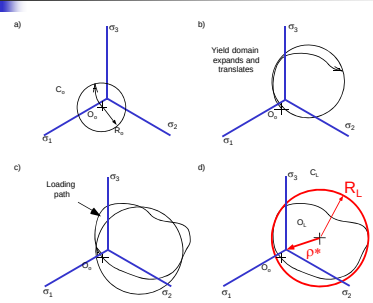


Isotropic Hardening



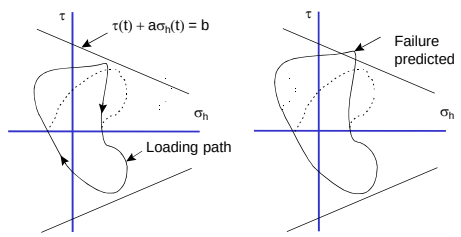
Failure occurs when the stress range is not elastic

Multiaxial Kinematic and Isotropic



ρ* stabilized residual stress

Dang Van (continued)



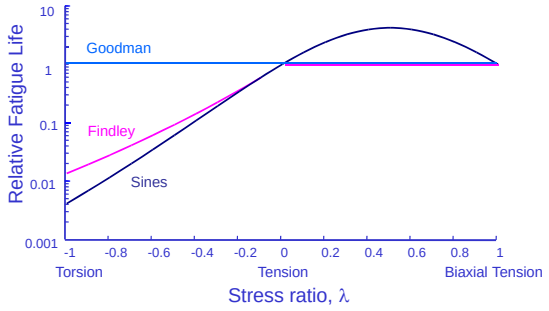
Stress Based Models Summary

Sines: $\frac{\Delta\tau_{oct}}{2} + \alpha(3\sigma_h) = \beta$

Findley: $\left(\frac{\Delta\tau}{2} + k\sigma_n\right)_{max} = f$

Dang Van: $\tau(t) + a\sigma_h(t) = b$

Model Comparison R = -1



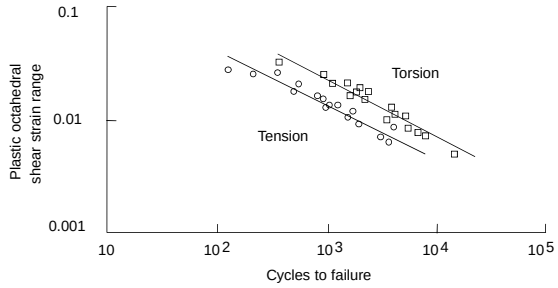
Outline

- State of Stress
- Stress-Strain Relationships
- Fatigue Mechanisms
- Multiaxial Testing
- Stress Based Models
- **Strain Based Models**
- Fracture Mechanics Models
- Nonproportional Loading
- Stress Concentrations

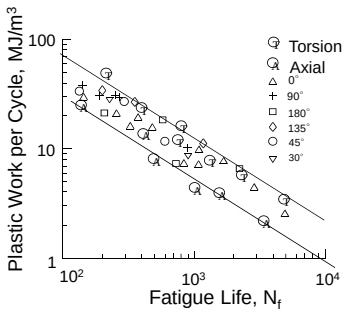
Strain Based Models

- Plastic Work
- Brown and Miller
- Fatemi and Socie
- Smith Watson and Topper
- Liu

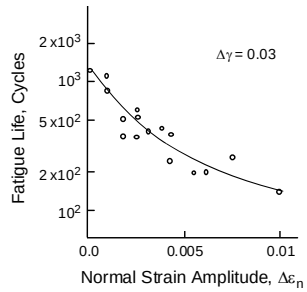
Octahedral Shear Strain



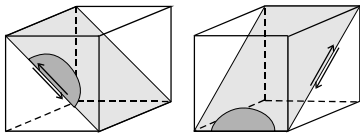
Plastic Work



Brown and Miller



Case A and B



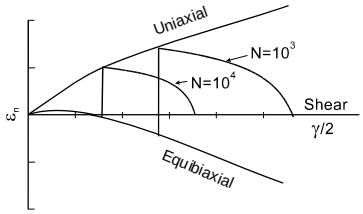
Case A

Growth along the surface

Case B

Growth into the surface

Brown and Miller (continued)

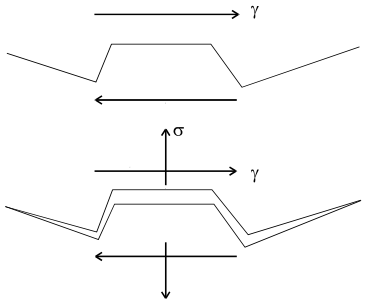


Brown and Miller (continued)

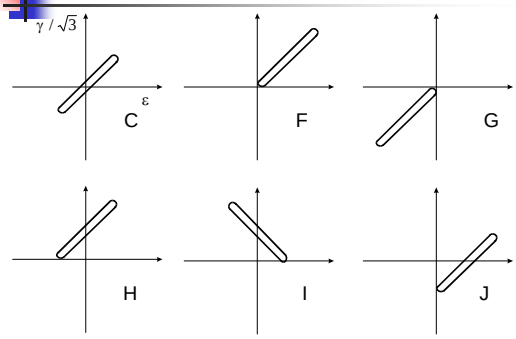
$$\Delta\hat{\gamma} = \left(\Delta\gamma_{max}^{\alpha} + S\Delta\varepsilon_n^{\alpha}\right)^{\frac{1}{\alpha}}$$

$$\frac{\Delta\gamma_{max}}{2} + S\Delta\varepsilon_n = A\frac{\sigma_f' - 2\sigma_{n,mean}}{E}(2N_f)^b + B\varepsilon_f'(2N_f)^c$$

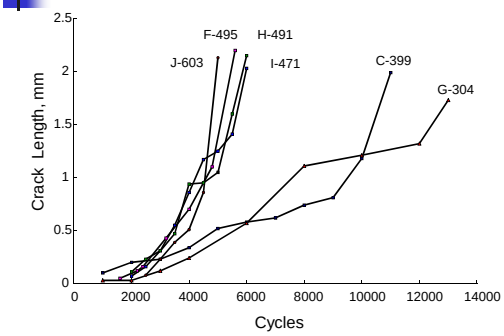
Fatemi and Socie



Loading Histories



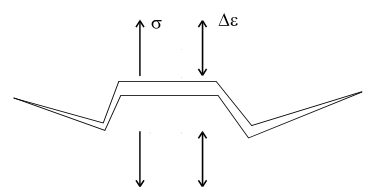
Crack Length Observations



Fatemi and Socie

$$\frac{\Delta\gamma}{2}\left(1+k\frac{\sigma_{n,max}}{\sigma_y}\right)=\frac{\tau_f'}{G}(2N_f)^{bo}+\gamma_f'(2N_f)^{co}$$

Smith Watson Topper



SWT

$$\sigma_n\frac{\Delta\varepsilon_1}{2}=\frac{\sigma_f'^2}{E}(2N_f)^{2b}+\sigma_f'\varepsilon_f'(2N_f)^{b+c}$$

Liu

Virtual strain energy for both mode I and mode II cracking

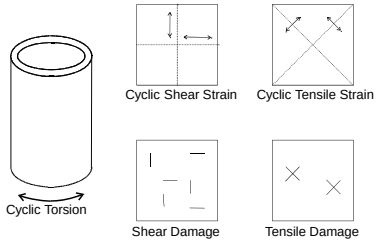
$$\Delta W_I = (\Delta\sigma_n\,\Delta\varepsilon_n)_{max} + (\Delta\tau\,\Delta\gamma)$$

$$\Delta W_I = 4\sigma_f'\varepsilon_f'(2N_f)^{b+c} + \frac{4\sigma_f'^2}{E}(2N_f)^{2b}$$

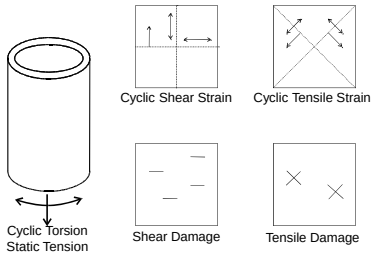
$$\Delta W_{II} = (\Delta\sigma_n\,\Delta\varepsilon_n) + (\Delta\tau\,\Delta\gamma)_{max}$$

$$\Delta W_{II} = 4\tau_f'\gamma_f'(2N_f)^{bo+co} + \frac{4\tau_f'^2}{G}(2N_f)^{2bo}$$

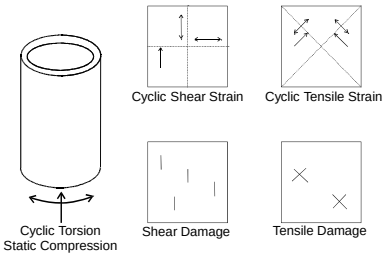
Cyclic Torsion



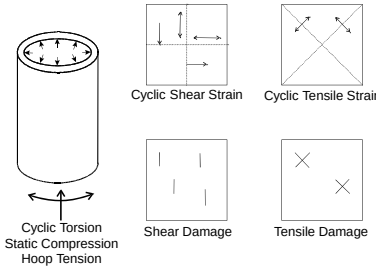
Cyclic Torsion with Static Tension



Cyclic Torsion with Compression



Cyclic Torsion with Tension and Compression



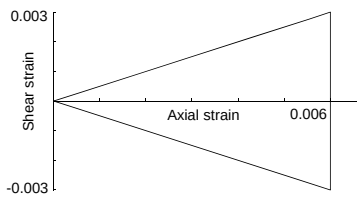
Test Results

Load Case	$\Delta\gamma/2$	σ_{hoop} MPa	σ_{axial} MPa	N_f
Torsion	0.0054	0	0	45,200
with tension	0.0054	0	450	10,300
with compression	0.0054	0	-500	50,000
with tension and compression	0.0054	450	-500	11,200

Conclusions

- All critical plane models correctly predict these results
- Hydrostatic stress models can not predict these results

Loading History



Model Comparison

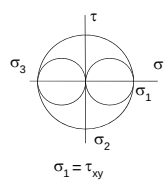
Summary of calculated fatigue lives

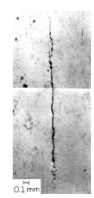
Model	Equation	Life
Epsilon	6.5	14,060
Garud	6.7	5,210
Ellyin	6.17	4,450
Brown-Miller	6.22	3,980
SWT	6.24	9,930
Liu I	6.41	4,280
Liu II	6.42	5,420
Chu	6.37	3,040
Gamma		26,775
Fatemi-Socie	6.23	10,350
Glinka	6.39	33,220

Strain Based Models Summary

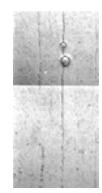
- Two separate models are needed, one for tensile growth and one for shear growth
- Cyclic plasticity governs stress and strain ranges
- Mean stress effects are a result of crack closure on the critical plane

Separate Tensile and Shear Models

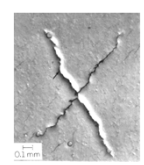




Inconel



1045 steel



stainless steel

Cyclic Plasticity

$\Delta\varepsilon$

$\Delta\gamma$

$\Delta\varepsilon^p$

$\Delta\gamma^p$

$\Delta\varepsilon\Delta\sigma$

$\Delta\gamma\Delta\tau$

$\Delta\varepsilon^p\Delta\sigma$

$\Delta\gamma^p\Delta\tau$

Mean Stresses

$$\Delta\varepsilon_{eq} = \frac{\sigma_f' - \sigma_{mean}}{E}(2N_f)^b + \varepsilon_f'(2N_f)^c$$

$$\frac{\Delta\gamma_{max}}{2} + S\Delta\varepsilon_n = (1.3 + 0.7S)\frac{\sigma_f' - 2\sigma_n}{E}(2N_f)^b + (1.5 + 0.5S)\varepsilon_f'(2N_f)^c$$

$$\frac{\Delta\gamma}{2}\left(1 + k\frac{\sigma_{n,max}}{\sigma_y}\right) = \frac{\tau_f'}{G}(2N_f)^{bo} + \gamma_f'(2N_f)^{co}$$

$$\sigma_n\frac{\Delta\varepsilon_1}{2} = \frac{\sigma_f'^2}{E}(2N_f)^{2b} + \sigma_f'\varepsilon_f'(2N_f)^{b+c}$$

$$\Delta W_i = \left[(\Delta\sigma_n\Delta\varepsilon_n)_{max} + (\Delta\tau\Delta\gamma)\right]\left(\frac{2}{1-R}\right)$$

Outline

- State of Stress
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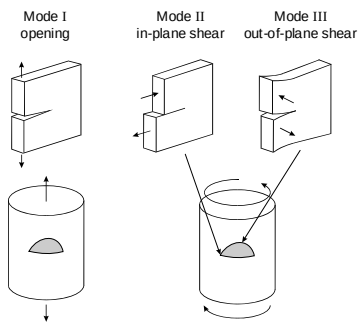
- Fatigue Mechanisms
- Multiaxial Testing
- Stress Based Models
- Strain Based Models
- **Fracture Mechanics Models**
- Nonproportional Loading
- Stress Concentrations

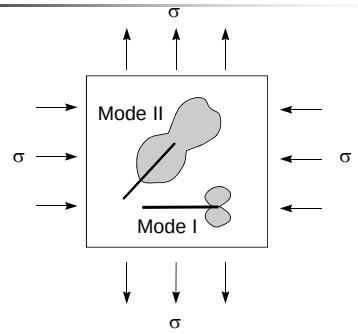
Fracture Mechanics Models

- Mode I growth
- Torsion
- Mode II growth
- Mode III growth

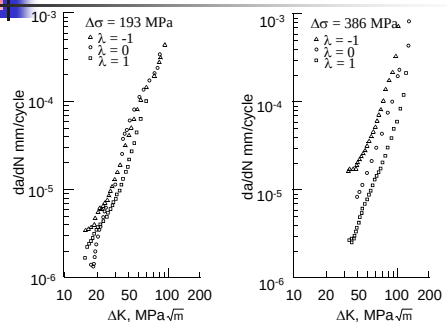
Mode I, Mode II, and Mode III



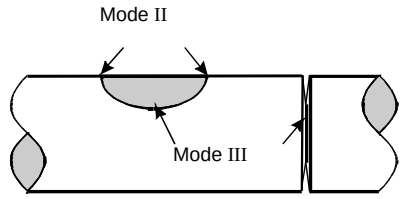
Mode I and Mode II Surface Cracks



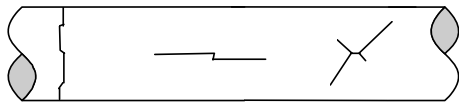
Biaxial Mode I Growth



Surface Cracks in Torsion

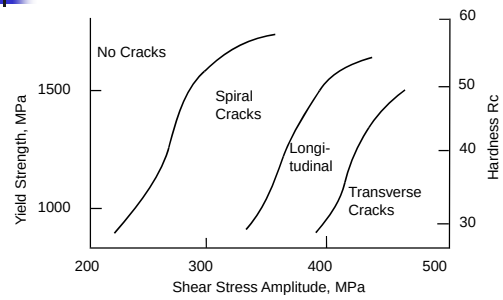


Failure Modes in Torsion

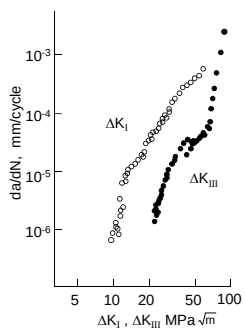


Transverse Longitudinal Spiral

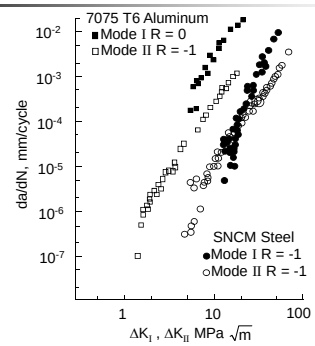
Fracture Mechanism Map



Mode I and Mode III Growth



Mode I and Mode II Growth



Fracture Mechanics Models

$$\frac{da}{dN} = C(\Delta K_{eq})^m$$

$$\Delta K_{eq} = \left[\Delta K_I^4 + 8\Delta K_{II}^4 + 8\Delta K_{III}^4/(1-\nu)\right]^{0.25}$$

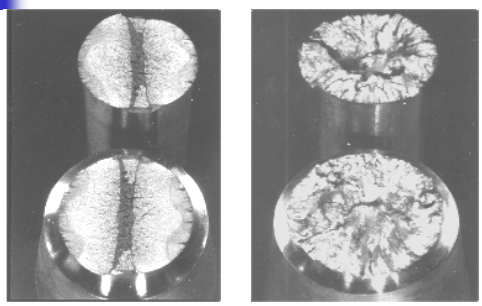
$$\Delta K_{eq} = \left[\Delta K_I^2 + \Delta K_{II}^2 + (1+\nu)\Delta K_{III}^2\right]^{0.5}$$

$$\Delta K_{eq} = \left[\Delta K_I^2 + \Delta K_I \Delta K_{II} + \Delta K_{II}^2\right]^{0.5}$$

$$\Delta K_{eq}(\varepsilon) = \left[(F_{II}\frac{E}{2(1+\nu)}\Delta\gamma)^2 + (F_I E\Delta\varepsilon)^2\right]^{0.5}\sqrt{\pi a}$$

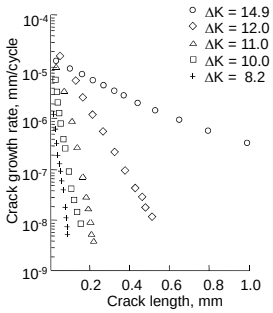
$$\Delta K_{eq}(\varepsilon) = FG\Delta\gamma\left(1 + k\frac{\sigma_{n,max}}{\sigma_{ys}}\right)\sqrt{\pi a}$$

Fracture Surfaces

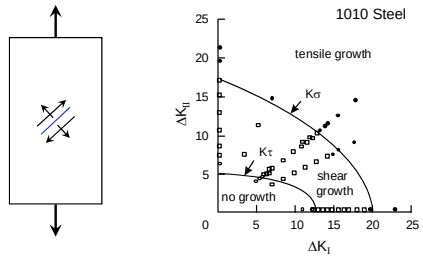


Bending Torsion

Mode III Growth



Growth of Inclined Cracks



Cracks grow in either tension or shear

From: Otsuka et.al. *Engineering Fracture Mechanics*, Vol 7, 1975

Otsuka

Tensile growth:

$$K\sigma = \cos\frac{\theta}{2}\left[K_I \cos^2\frac{\theta}{2} - \frac{3}{2}K_{II}\sin\theta\right]$$

Shear growth:

$$K\tau = \frac{1}{2}\cos\frac{\theta}{2}\left[K_I \sin\frac{\theta}{2} + K_{II}(3\cos\theta - 1)\right]$$

Strain Energy Density

Strain energy density at the crack tip:

$$S = a_{11}K_I^2 + 2a_{12}K_I K_{II} + a_{22}K_{II}^2 + a_{33}K_{III}^2$$

Necessary and sufficient conditions for crack growth:

$$\frac{\partial S}{\partial \theta} = 0 \quad \text{at } \theta = \theta_o$$

$$\frac{\partial^2 S}{\partial \theta^2} > 0 \quad \text{at } \theta = \theta_o$$

Cyclic strain energy density:

$$\Delta S = 2[a_{11}(\theta_o)K_I^{mean}\Delta K_I + a_{12}(\theta_o)(K_{II}^{mean}\Delta K_I + K_I^{mean}\Delta K_{II}) + a_{22}(\theta_o)K_{II}^{mean}\Delta K_{II} + a_{33}(\theta_o)K_{III}^{mean}\Delta K_{III}]$$

Sih, G.C and Barthelemy, B.M. "Mixed Mode Fatigue Crack Growth Predictions" *Engineering Fracture Mechanics*, Vol. 13, 1980

Fracture Mechanics Models Summary

- Multiaxial loading has little effect in Mode I
- Crack closure makes Mode II and Mode III calculations difficult

Outline

- State of Stress
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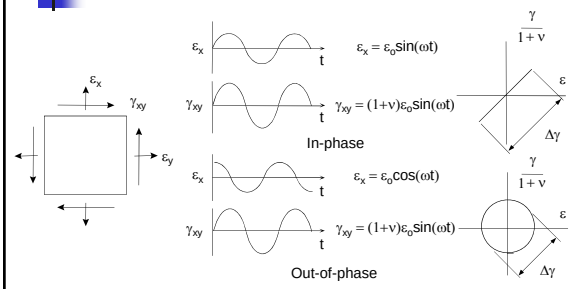
- Stress Concentrations

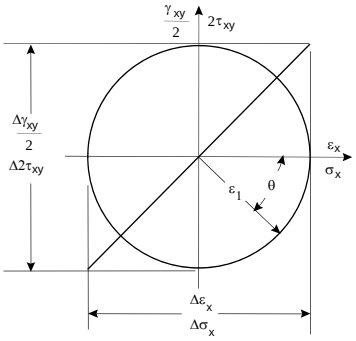
Nonproportional Loading

- In and Out-of-phase loading
- Nonproportional cyclic hardening
- Variable amplitude

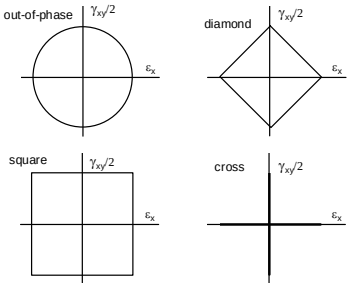
In and Out-of-Phase Loading



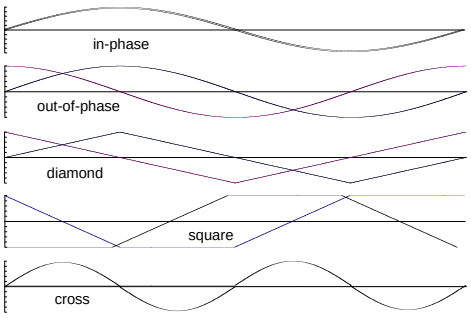
In-Phase and Out-of-Phase



Loading Histories

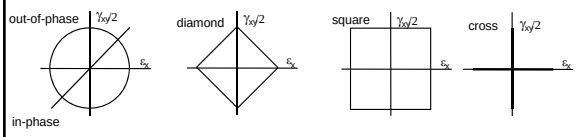


Loading Histories

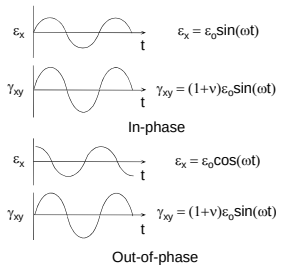


Findley Model Results

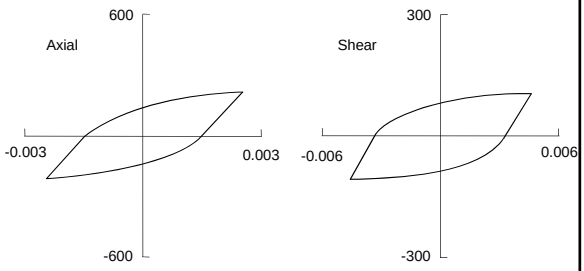
	$\Delta\tau/2$ MPa	$\sigma_{n,max}$ MPa	$\Delta\tau/2 + 0.3\,\sigma_{n,max}$	N/N_{ip}
in-phase	353	250	428	1.0
90° out-of-phase	250	500	400	2.0
diamond	250	500	400	2.0
square	353	603	534	0.11
cross - tension cycle	250	250	325	16
cross - torsion cycle	250	0	250	216



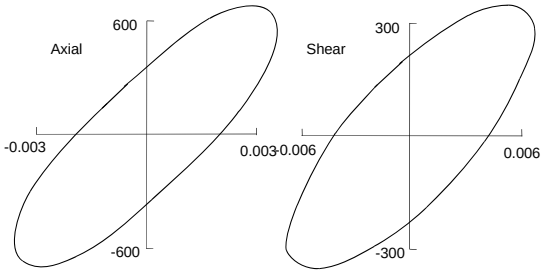
Nonproportional Hardening



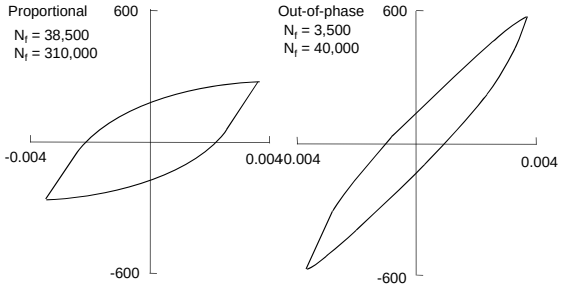
In-Phase



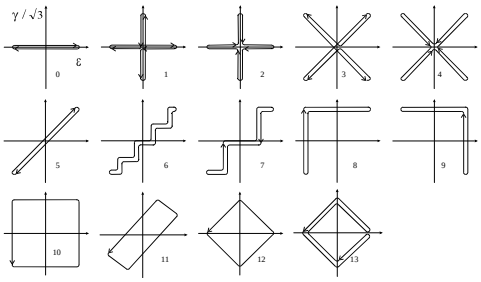
90° Out-of-Phase



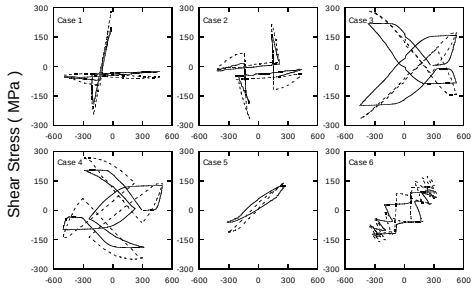
Critical Plane



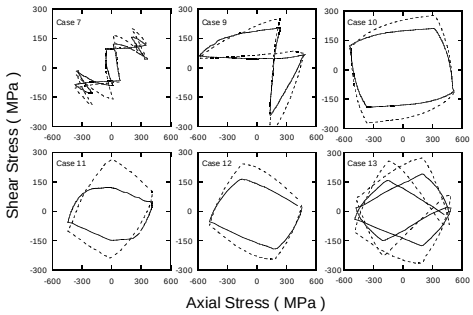
Loading Histories



Stress-Strain Response

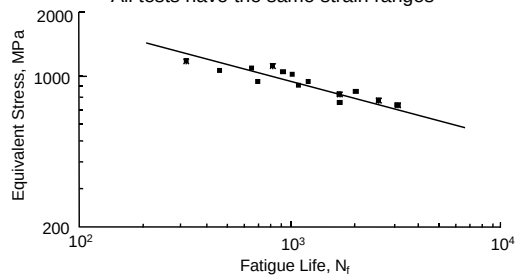


Stress-Strain Response (continued)



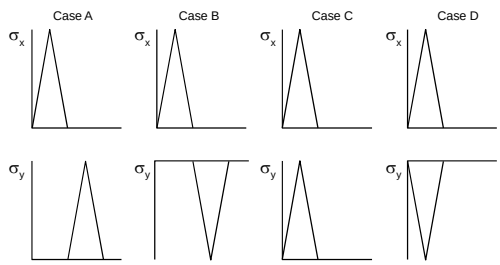
Maximum Stress

All tests have the same strain ranges

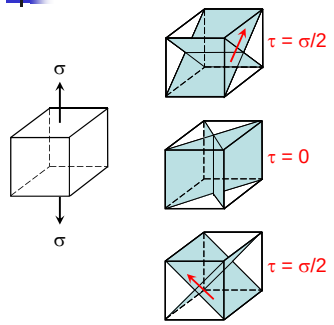


Nonproportional hardening results in lower fatigue lives

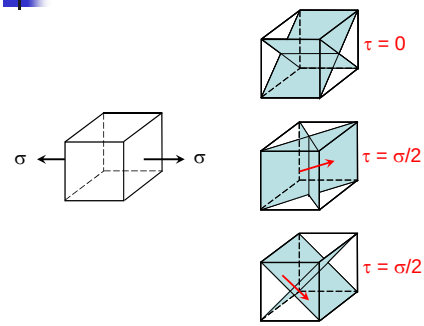
Nonproportional Example



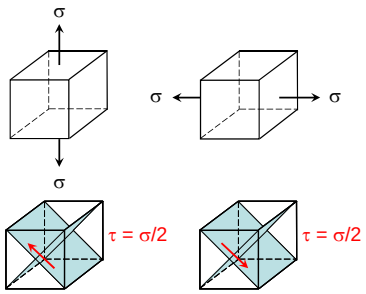
Shear Stresses on Planes



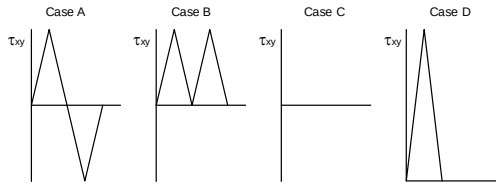
Shear Stresses on Planes



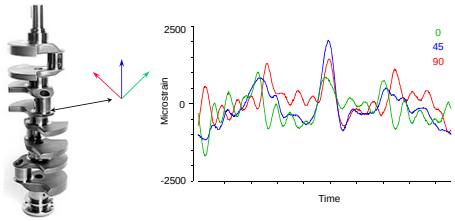
Combined



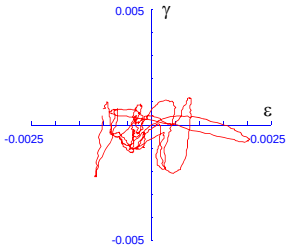
Shear Stresses



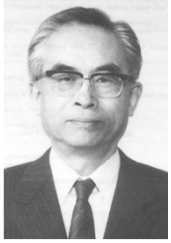
Variable Amplitude

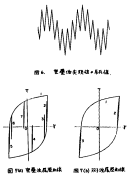


Cycle Counting and Damage?



Rainflow Cycle Counting

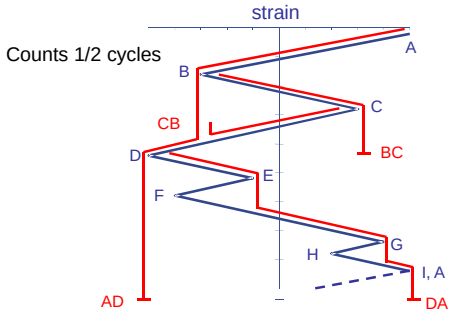




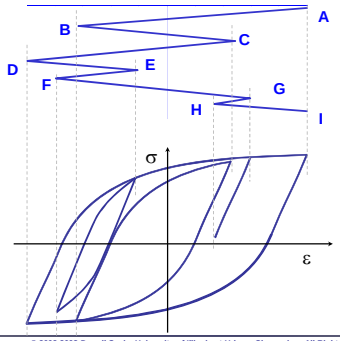
What could be more basic than learning to count correctly?

Matsuishi and Endo (1968) Fatigue of Metals Subjected to Varying Stress – Fatigue Lives Under Random Loading, Proceedings of the Kyushu District Meeting, JSME, 37-40

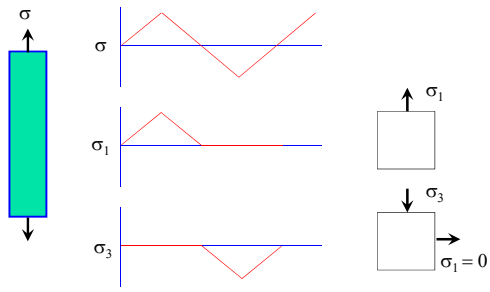
Rainflow



Rainflow and Hysteresis



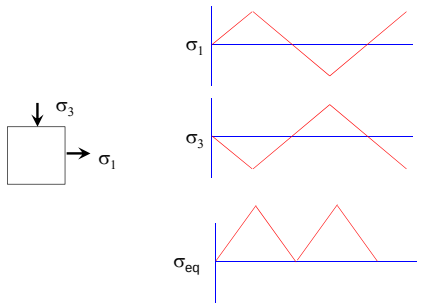
Principal Stress Directions



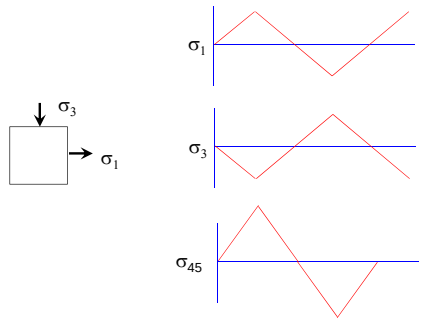
Signed Equivalent Stress

$$\sigma_{eq} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2}\ \mathrm{sgn}\left(|\sigma|_{max}\right)$$

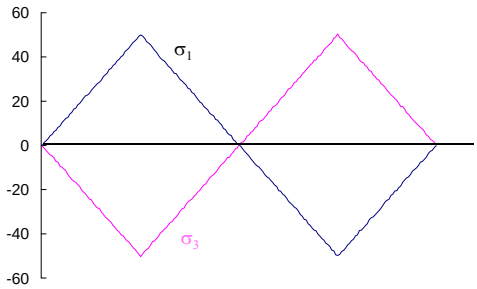
Torsion Stresses



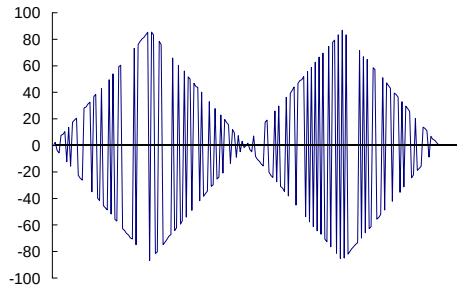
Torsion Stresses



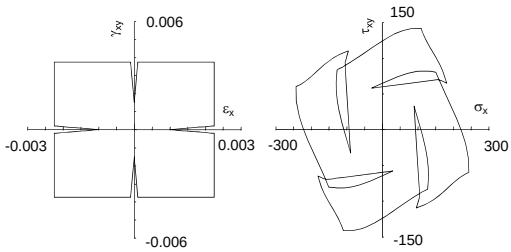
Real Data



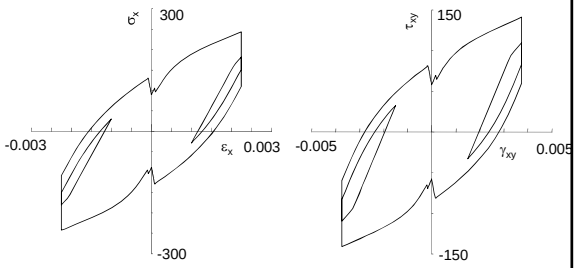
Signed Equivalent Stress



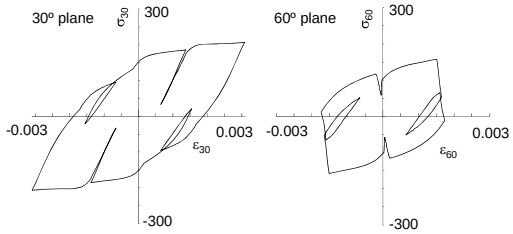
Simple Variable Amplitude History



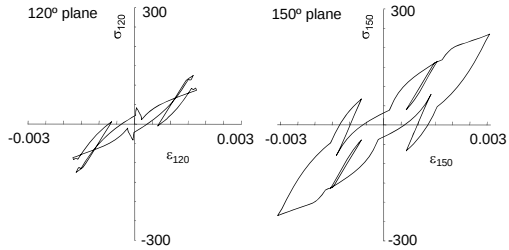
Stress-Strain on 0° Plane



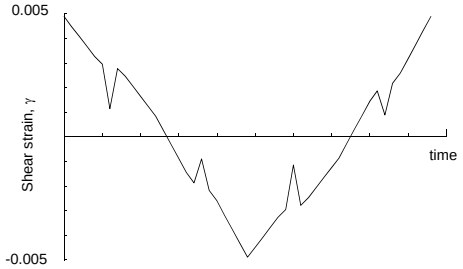
Stress-Strain on 30° and 60° Planes



Stress-Strain on 120° and 150° Planes



Shear Strain History on Critical Plane



Fatigue Calculations

Load or strain history

↘

Cyclic plasticity model

↘

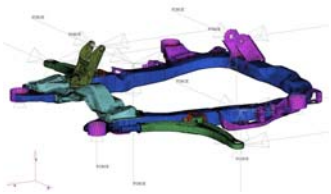
Stress and strain tensor

↘

Search for critical plane

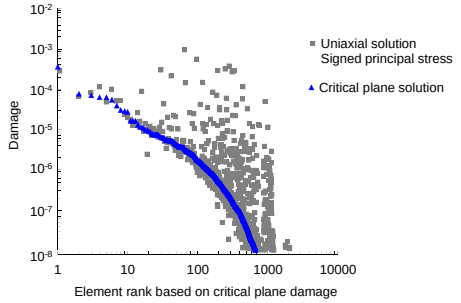
An Example

- Analysis model
  Single event
  16 input channels
  2240 elements



From Khosrovaneh, Pattu and Schnaidt "Discussion of Fatigue Analysis Techniques for Automotive Applications" Presented at SAE 2004.

Biaxial and Uniaxial Solution



Nonproportional Loading Summary

- Nonproportional cyclic hardening increases stress levels
- Critical plane models are used to assess fatigue damage

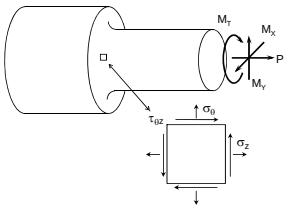
Outline

- State of Stress
- Stress-Strain Relationships
- Fatigue Mechanisms
- Multiaxial Testing
- Stress Based Models
- Strain Based Models
- Fracture Mechanics Models
- Nonproportional Loading
- **Stress Concentrations**

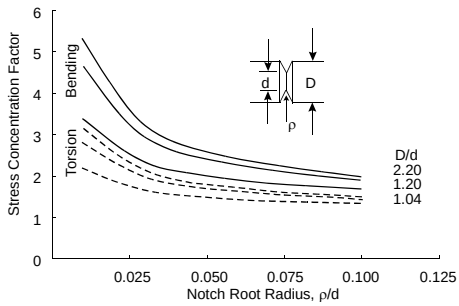
Notches

- Stress and strain concentrations
- Nonproportional loading and stressing
- Fatigue notch factors
- Cracks at notches

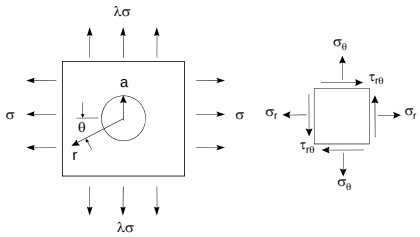
Notched Shaft Loading



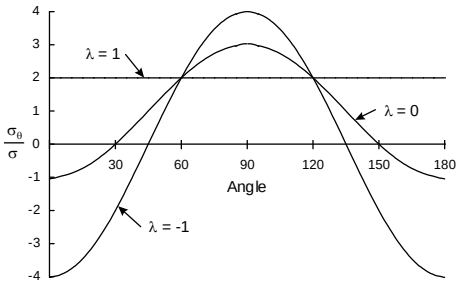
Stress Concentration Factors



Hole in a Plate

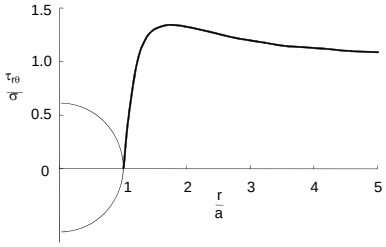


Stresses at the Hole

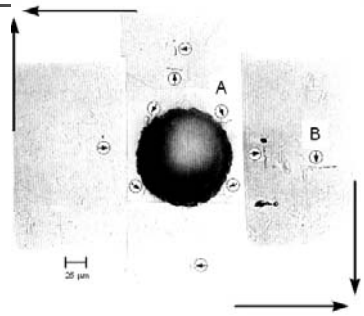


Stress concentration factor depends on type of loading

Shear Stresses during Torsion



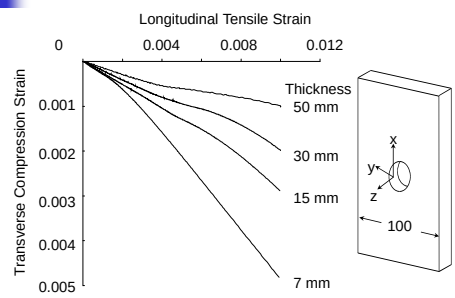
Torsion Experiments



Multiaxial Loading

- Uniaxial loading that produces multiaxial stresses at notches
- Multiaxial loading that produces uniaxial stresses at notches
- Multiaxial loading that produces multiaxial stresses at notches

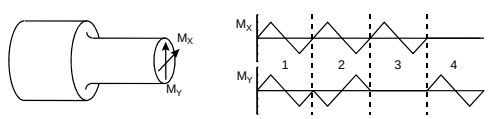
Thickness Effects



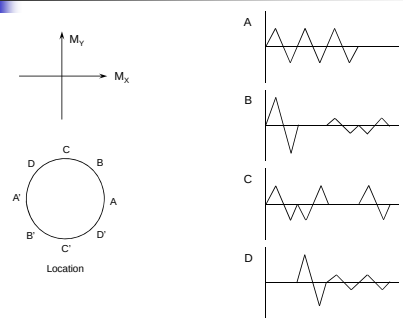
Multiaxial Loading

- Uniaxial loading that produces multiaxial stresses at notches
- Multiaxial loading that produces uniaxial stresses at notches
- Multiaxial loading that produces multiaxial stresses at notches

Applied Bending Moments



Bending Moments on the Shaft



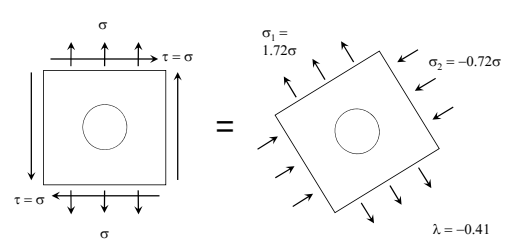
Bending Moments

ΔM	A	B	C	D
2.82		1		1
2.00	3		2	
1.41		2		1
1.00			2	
0.71				2

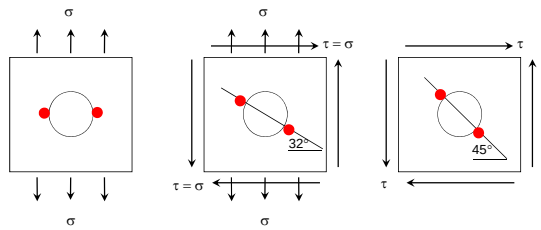
$$\Delta\overline{M} = \sqrt[5]{\sum \Delta M^5}$$

	A	B	C	D
$\Delta\overline{M}$	2.49	2.85	2.31	2.84

Combined Loading



Maximum Tensile Stress Location

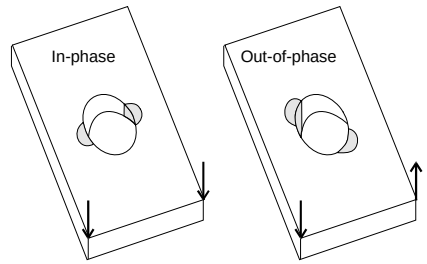


$K_t = 3$, $\sigma_1 = \sigma$

$K_t = 3.41$, $\sigma_1 = 1.72\sigma$

$K_t = 4$, $\sigma_1 = \tau$

In and Out of Phase Loading



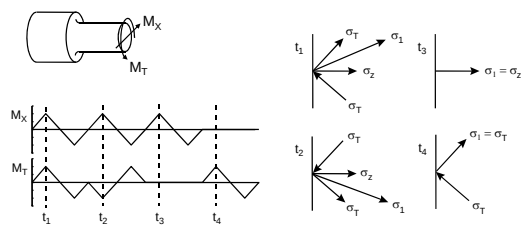
$K_t = 3$ $K_t = 4$

Damage location changes with load phasing

Multiaxial Loading

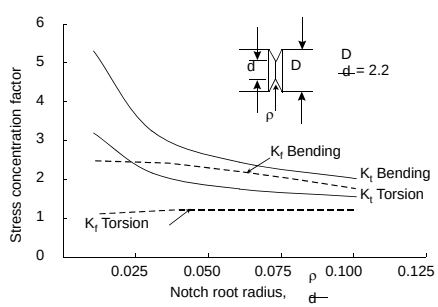
- Uniaxial loading that produces multiaxial stresses at notches
- Multiaxial loading that produces uniaxial stresses at notches
- Multiaxial loading that produces multiaxial stresses at notches

Torsion Loading

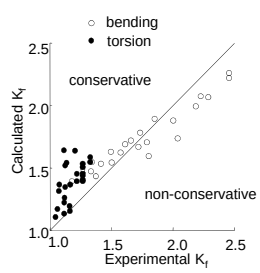


Out-of-phase shear loading is needed to produce nonproportional stressing

Fatigue Notch Factors



Fatigue Notch Factors (continued)



Peterson's Equation

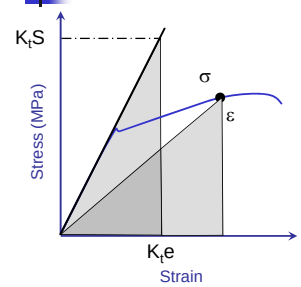
$$K_f = 1 + \frac{K_T - 1}{1 + \frac{a}{r}}$$

Fracture Surfaces in Torsion

Circumferencial Notch

Shoulder Fillet

Neuber's Rule



Actual stress

$$\underbrace{K_t S}\, K_t e = \sigma\varepsilon$$

Stress calculated with elastic assumptions

$$^eS\ ^ee = \sigma\varepsilon$$

For cyclic loading

$$\Delta^eS^2 = E\Delta\sigma\Delta\varepsilon$$

Multiaxial Neuber's Rule

Define Neuber's rule in equivalent variables

$$\Delta^e\bar{S}^2 = E\Delta\bar{\sigma}\Delta\bar{\varepsilon}$$

Stress strain curve

$$\Delta\bar{\varepsilon} = \frac{\Delta\bar{\sigma}}{E} + \left(\frac{\Delta\bar{\sigma}}{K'}\right)^{\frac{1}{n'}}$$

Constitutive equation

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix} = \begin{bmatrix} & & \\ & f(E,K',n') & \\ & & \end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix}$$

Five equations and six unknowns

Ignore Plasticity Theory

$$\varepsilon_2 = \frac{^ee_2}{^ee_1}\varepsilon_1$$

$$\varepsilon_3 = \frac{^ee_3}{^ee_1}\varepsilon_1$$

$$\sigma_2 = \frac{^eS_2}{^eS_1}\sigma_1$$

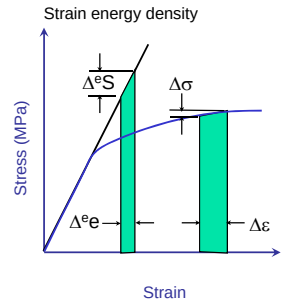
$$\sigma_3 = \frac{^eS_3}{^eS_1}\sigma_1$$

Hoffman and Seeger

$$\frac{\sigma_2}{\sigma_1} = \frac{^eS_2}{^eS_1}$$

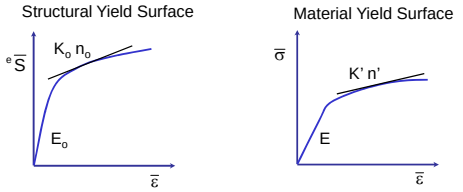
$$\frac{\varepsilon_2}{\varepsilon_1} = \frac{^ee_2}{^ee_1}$$

Glinka



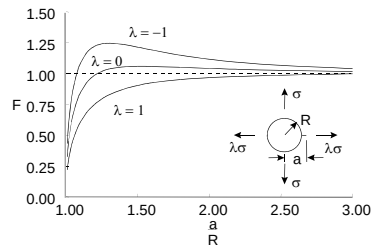
$$\frac{\Delta\sigma_{ij}\Delta\varepsilon_{ij}}{\sum\Delta\sigma_{ij}\Delta\varepsilon_{ij}} = \frac{\Delta^eS_{ij}\Delta^ee_{ij}}{\sum\Delta^eS_{ij}\Delta^ee_{ij}}$$

Koettgen-Barkey-Socie



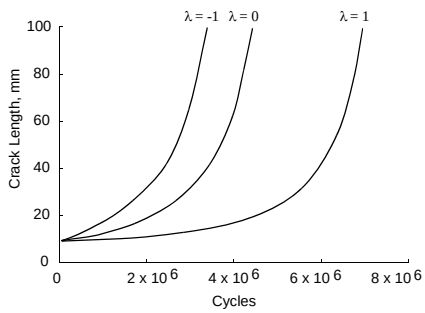
$$\begin{bmatrix} {}^{e}S_x \\ {}^{e}S_y \\ {}^{e}T_{xy} \end{bmatrix} = \begin{bmatrix} f_o(E_o,K_o,n_o) \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \qquad \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} f(E,K',n') \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

Stress Intensity Factors



$$\frac{da}{dN} = C\left(\Delta K_{eq}\right)^m \qquad K_I = F\Delta\sigma\sqrt{\pi a}$$

Crack Growth From a Hole



Notches Summary

- Uniaxial loading can produce multiaxial stresses at notches
- Multiaxial loading can produce uniaxial stresses at notches
- Multiaxial stresses are not very important in thin plate and shell structures
- Multiaxial stresses are not very important in crack growth

Multiaxial Fatigue