Frequency Based Fatigue

Professor Darrell F. Socie
Mechanical Engineering
University of Illinois at Urbana-Champaign

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Deterministic versus Random

Deterministic – from past measurements the future position of a satellite can be predicted with reasonable accuracy

Random – from past measurements the future position of a car can only be described in terms of probability and statistical averages
Frequency Domain

Strain (ustrain)

Frequency (Hz)
Statistics of Time Histories

- Mean or Expected Value
- Variance / Standard Deviation
- Coefficient of Variation
- Root Mean Square
- Kurtosis
- Skewness
- Crest Factor
- Irregularity Factor
Mean or Expected Value

Central tendency of the data

Mean $= \mu_x = \bar{x} = E(X) = \frac{\sum_{i=1}^{N} x_i}{N}$
Variance / Standard Deviation

Dispersion of the data

\[
\text{Var}(X) = \frac{\sum_{i=1}^{N} (x_i - \bar{x})^2}{N}
\]

Standard deviation

\[
\sigma_x = \sqrt{\text{Var}(X)}
\]
Coefficient of Variation

\[ \text{COV} = \frac{\sigma_x}{\mu_x} \]

Useful to compare different dispersions

\[
\begin{align*}
\mu &= 10 & \mu &= 100 \\
\sigma &= 1 & \sigma &= 10 \\
\text{COV} &= 0.1 & \text{COV} &= 0.1
\end{align*}
\]
Root Mean Square

\[ \text{RMS} = \sqrt{\frac{\sum_{i=1}^{N} x_i^2}{N}} \]

The rms is equal to the standard deviation when the mean is 0
Skewness

Skewness is a measure of the asymmetry of the data around the sample mean. If skewness is negative, the data are spread out more to the left of the mean than to the right. If skewness is positive, the data are spread out more to the right. The skewness of the normal distribution (or any perfectly symmetric distribution) is zero.

\[ \text{Skewness}(X) = \frac{\sum_{i=1}^{N} (x_i - \bar{x})^3}{N\sigma^3} \]
Kurtosis is a measure of how outlier-prone a distribution is. The kurtosis of the normal distribution is 3. Distributions that are more outlier-prone than the normal distribution have kurtosis greater than 3; distributions that are less outlier-prone have kurtosis less than 3.

\[
\text{Kurtosis}(X) = \frac{\sum_{i=1}^{N} (X_i - \bar{X})^4}{N\sigma^4}
\]
Crest Factor

The crest factor is the ratio of the peak (maximum) value to the root-mean-square (RMS) value. A sine wave has a crest factor of 1.414.
Irregularity Factor

- Positive zero crossing
- Peak

\[
IF = \frac{E(0^+)}{E(P)} = \frac{4}{7}
\]

IF \rightarrow 1 is narrow band signal
IF \rightarrow 0 is wide band signal
Time Domain Nomenclature

- Random
- Stochastic
- Stationary
- Non-stationary
- Gaussian
- Narrow-band
- Wide-band
The instantaneous value can not be predicted at any future time.
Stochastic processes provide suitable models for physical systems where the phenomena is governed by probabilities.
Stationary

The properties computed over short time intervals, $t + \Delta t$, do not significantly vary from each other.
Non-stationary
Gaussian

Normally distributed around the mean
Narrow-band
Wide-band
Frequency Domain Nomenclature

Fourier
Nyquist frequency
Aliasing
Anit-aliasing filter
FFT
Inverse FFT
Autospectral density
Transfer Function
Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.
Fourier Series

\[ X(t) = a_0 + \sum_{k=1}^{\infty} a_k \sin(k\omega_0 t) + b_k \cos(k\omega_0 t) \]

\[ X(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_0 t + \theta_k) \]

\[ X(t) = \sum_{k=-\infty}^{\infty} X_S[k] e^{jk\omega_0 t} \]

\( a_k, b_k, c_k, X_S[k] \) are Fourier coefficients

frequency, magnitude, and phase are all described by the coefficients
What $\Delta t$ should be used for sampling the data?
Aliasing

2 signals that differ in frequency by a factor of 2

Both signals have the same digital samples.
Nyquist frequency

Real spectrum

Aliased spectrum

\[ f_c = \frac{1}{2\Delta t} \]

Sample at 2 times the frequency of interest
Anti-aliasing Filter

Phase

Magnitude, dB

frequency
Fourier Transform

The area under each spike represents the magnitude of the sine wave at that frequency.

The magnitude of the FFT depends on the frequency window \( \Delta f \).
Fast Fourier Transform - FFT

\[ X(t) = c_0 + \sum_{k=1}^{N/2} c_k \cos(k \omega_0 t + \theta_k) \]
Fast Fourier Transform - FFT

\[ X(t) = c_0 + \sum_{k=1}^{N/2} c_k \cos(k\omega_0 t + \theta_k) \]
Inverse FFT

\[
X(t) = c_o + \sum_{k=1}^{N/2} c_k \cos(k\omega_o t + \theta_k)
\]

c_k, \theta_k, and k\omega_o are all known
## Autospectral Density

<table>
<thead>
<tr>
<th>Function</th>
<th>Units</th>
<th>Formula/Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time History</td>
<td>EU</td>
<td>$X(n)$</td>
</tr>
<tr>
<td>Linear Spectrum</td>
<td>EU</td>
<td>$S(n) = \text{DFT}(X(n))$</td>
</tr>
<tr>
<td>AutoPower</td>
<td>EU^2</td>
<td>$AP(n) = S(n) \cdot S(n)$</td>
</tr>
<tr>
<td>PSD</td>
<td>(EU^2)/Hz</td>
<td>$\text{PSD}(n) = \frac{AP(n)}{(W_f \cdot \Delta f)}$</td>
</tr>
<tr>
<td>ESD</td>
<td>(EU^2*sec)/Hz</td>
<td>$\text{ESD}(n) = \frac{AP(n) \cdot T}{(W_f \cdot \Delta f)}$</td>
</tr>
</tbody>
</table>
Calculating Autospectral Density

\[ S(f_k) = \frac{1}{n_D N_D \Delta t} \sum_{i=1}^{n_D} |X(f_k)|^2 \]

Magnitude only no phase information
Power Spectral Density - PSD

Average power associated with a 1 Hz frequency window centered at each frequency, $f$. Phase information is lost.
Linear Spectrum

Sometimes called Amplitude Spectral Density
<table>
<thead>
<tr>
<th>Frequency (Hz)</th>
<th>Strain (ustrain^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>10^1</td>
</tr>
<tr>
<td>10^5</td>
<td>10^2</td>
</tr>
<tr>
<td>10^4</td>
<td>10^3</td>
</tr>
<tr>
<td>10^3</td>
<td>10^4</td>
</tr>
<tr>
<td>10^2</td>
<td>10^5</td>
</tr>
<tr>
<td>10^1</td>
<td>10^6</td>
</tr>
</tbody>
</table>

**Comparison**

**Linear Spectrum**

**PSD**
Frequency Domain Limitations

Non-stationary signals

EASE1.EDT-Strain_c56

Strain (ustrain)

Time (Secs)
Linear Spectrum

![Linear Spectrum Diagram]

- Frequency (Hz)
- Strain (ustrain)

- Y-axis: \(10^{-1} \rightarrow 10^3\)
- X-axis: \(0 \rightarrow 140\)
Transfer Function

\[ \frac{\text{output}}{\text{input}} \]

Frequency (Hz)

\[ 10^0 \quad 10^1 \quad 10^2 \]

\[ 10^0 \quad 10^2 \quad 10^4 \]
Frequency Based Fatigue

- Stationary Loading
  - Wind
  - Sea State
  - Vibration
Assumptions

Random
Gaussian
Stationary
Fatigue Analysis

Load time history → Structural model → Stress time history

Time Domain

Rainflow → Miner linear → Durability damage

Frequency Domain

Acceleration PSD → Transfer function → Stress PSD

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Dynamics Model

\[ m \frac{d^2 y}{dt^2} = F(t) \]

\[ z(t) = y(t) - x(t) \]

\[ m \frac{d^2 z}{dt^2} + c \frac{dz}{dt} + k z = F(t) - m \frac{d^2 x}{dt^2} \]

\[ \frac{d^2 z}{dt^2} + 2\zeta \omega_n \frac{dz}{dt} + \omega_n^2 z = - \frac{d^2 x}{dt^2} \]
PSD Moments

\[ m_n = \sum_{k=1}^{N} f_k^n G_k(f_k) \delta f \]
Expected Values

zero crossings

$$E(0) = \sqrt{\frac{m_2}{m_0}}$$

peaks

$$E(P) = \sqrt{\frac{m_4}{m_2}}$$

Irregularity factor

$$IF = \frac{E(0)}{E(P)} = \sqrt{\frac{m_2^2}{m_0 m_4}}$$
The probability $P(\Delta S_i)$ of a stress range occurring between $\Delta S_i - \frac{\delta S}{2}$ and $\Delta S_i + \frac{\delta S}{2}$ is $P(\Delta S_i) = p(\Delta S_i)\delta S$. 

\[\Delta S_i - \frac{\delta S}{2} \quad \text{and} \quad \Delta S_i + \frac{\delta S}{2}\]
Fatigue Damage

Cycles at level $i \quad n_i = p(\Delta S_i) \delta S N_T$

Total cycles $N_T = E(P) T$

Fatigue life $N_f(\Delta S_i) = \left(\frac{\Delta S_i}{2S_f'}\right)^{\frac{1}{b}}$

Fatigue Damage $D = \sum_{i=1}^{N} \frac{n_i}{N_f(\Delta S_i)} = E(P) T \delta S (2S_f')^{\frac{1}{b}} \sum_{i=1}^{N} \frac{p(\Delta S_i)}{(\Delta S_i)^{\frac{1}{b}}}$

Fatigue damage is determined by $p(\Delta S_i)$
Narrow Band Solution

Rayleigh distribution

\[ p(\Delta S) = \frac{\Delta S}{4m_0} \exp\left[-\frac{\Delta S^2}{8m_0}\right] \]

IF \( \to 1 \) is narrow band signal

IF \( \to 0 \) is wide band signal

This wide band signal has the same peak distribution as this narrow band signal
Dirlik Solution

\[ p(\Delta S) = f( m_0, m_1, m_2, m_4 ) \]

\[ p(\Delta S) = \frac{D_1 \exp\left( -\frac{Z}{Q} \right) + D_2 \exp\left( -\frac{Z^2}{R^2} \right) + D_3 Z \exp\left( -\frac{Z^2}{2} \right)}{2\sqrt{m_0}} \]

\[ Z = \frac{\Delta S}{2\sqrt{m_0}} \quad IF = \frac{m_2}{\sqrt{m_0 m_4}} \quad X_m = \frac{m_1}{m_0} \sqrt{\frac{m_2}{m_4}} \quad D_1 = \frac{2(X_m - IF^2)}{1 + IF^2} \]

\[ R = \frac{IF - X_m - D_1^2}{1 - IF - D_1 + D_1^2} \quad D_2 = \frac{1 - IF - D_1 + D_1^2}{1 - R} \quad Q = \frac{5(IF - D_1 - D_2 R)}{4D_1} \]

\[ D_3 = 1 - D_1 - D_2 \]
Loading History
Rainflow Ranges

![Graph showing Rainflow Ranges](image)

The graph illustrates the relationship between stress range and the number of cycles for a PSD and a time history. The y-axis represents the number of cycles on a logarithmic scale, while the x-axis represents the stress range also on a logarithmic scale. The graph shows a decreasing trend as stress range increases.
Relative Fatigue Estimates

<table>
<thead>
<tr>
<th>SN Slope</th>
<th>Dirlik $\frac{N_{PSD}}{N_{TH}}$</th>
<th>Rayleigh $\frac{N_{PSD}}{N_{TH}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>371</td>
<td>109</td>
</tr>
<tr>
<td>5</td>
<td>6.6</td>
<td>1.9</td>
</tr>
<tr>
<td>3</td>
<td>1.8</td>
<td>0.5</td>
</tr>
</tbody>
</table>
Fatigue Data

\[\text{Damage} \propto \Delta S^n\]

Graph showing the relationship between amplitude and cycles for different values of \(n\): \(n = 10\), \(n = 5\), and \(n = 3\).
Loading History

Bracket.sif-Strain_b43

rf_000.sif-Strain_b43
Slope = 3
Slope = 5

![Graph showing damage distribution with axes labeled range (ustrain), mean (ustrain), and % damage. The graph has a 3D perspective with bars indicating the damage level. The damage range is from -750 to 750, and the % damage values range from 0 to 5.14.](image-url)
Slope = 10

Damage

% damage
0
20.78

range (ustrain)
1500 750

mean (ustrain)
-750

Damage
Frequency Based Fatigue