# Frequency Based Fatigue

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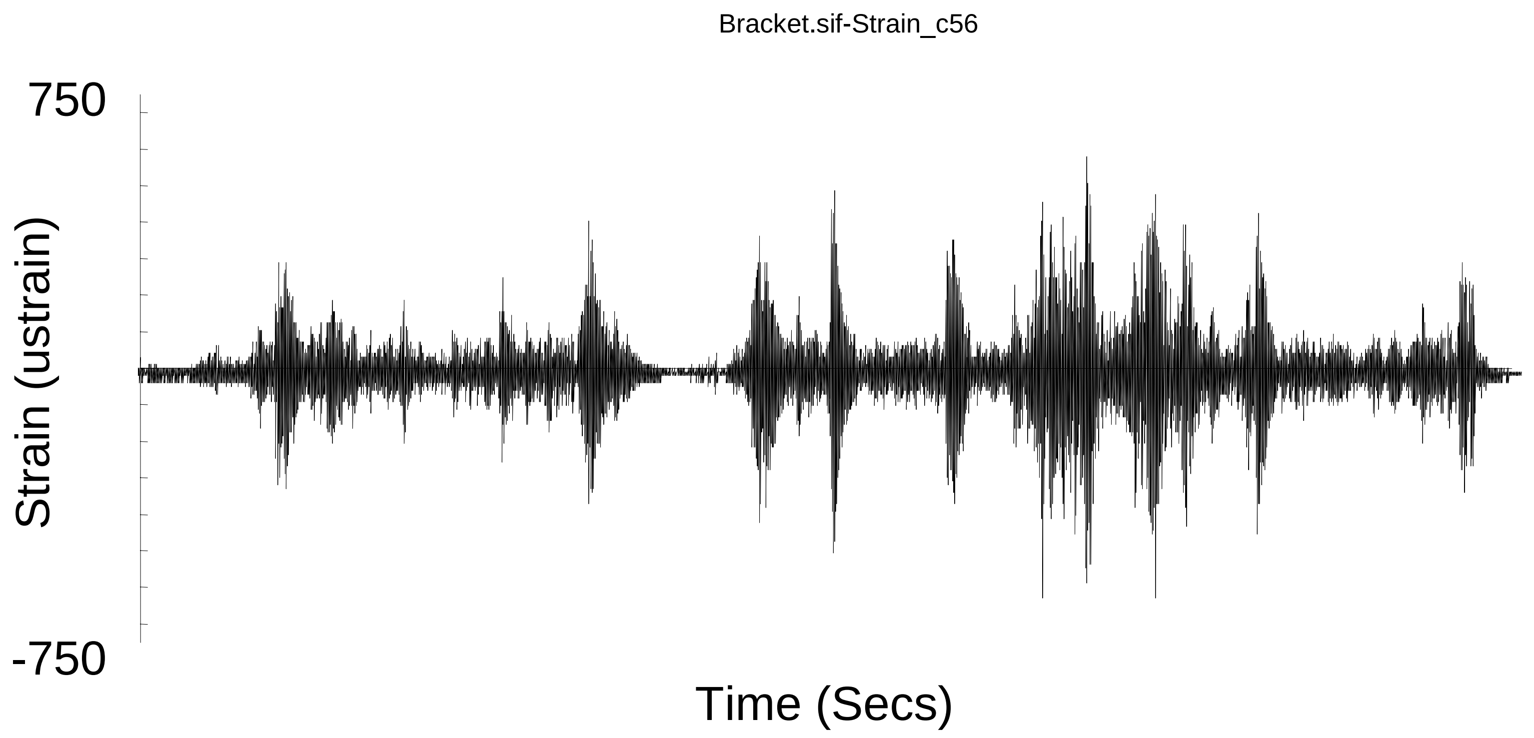
# Deterministic versus Random

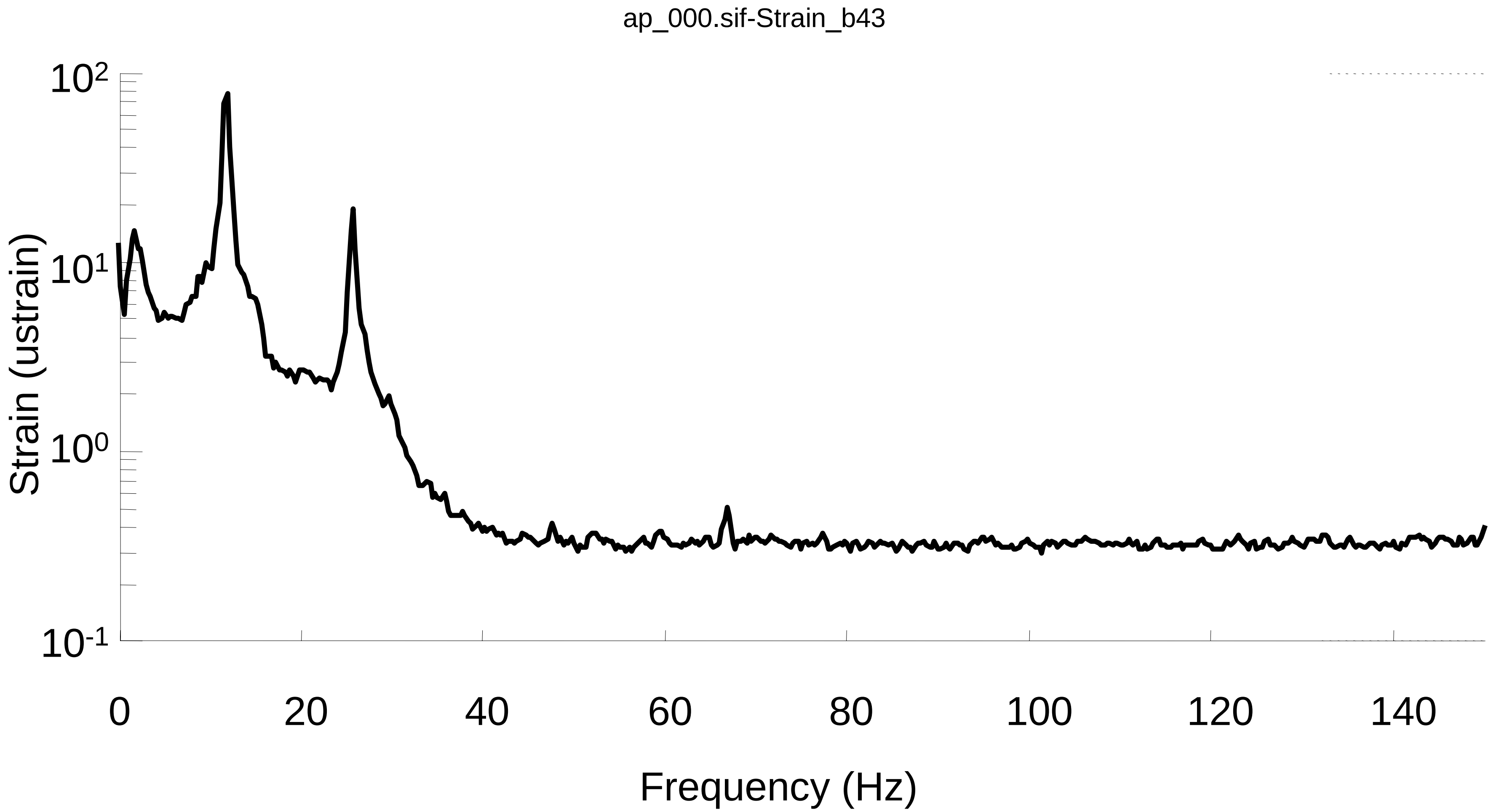
Deterministic – from past measurements the future position of a satellite can be predicted with reasonable accuracy

Random – from past measurements the future position of a car can only be described in terms of probability and statistical averages

# Time Domain



# Frequency Domain



# Statistics of Time Histories

- Mean or Expected Value
- Variance / Standard Deviation
- Coefficient of Variation
- Root Mean Square
- Kurtosis
- Skewness
- Crest Factor
- Irregularity Factor

# Mean or Expected Value

Central tendency of the data

$$\text{Mean} = \mu_x = \bar{x} = E(X) = \frac{\sum_{i=1}^{N} x_i}{N}$$

# Variance / Standard Deviation

Dispersion of the data

$$\mathrm{Var}(X) = \frac{\sum_{i=1}^{N} (x_i - \bar{x})^2}{N}$$

Standard deviation

$$\sigma_x = \sqrt{\mathrm{Var}(X)}$$

# Coefficient of Variation

$$COV = \frac{\sigma_x}{\mu_x}$$

Useful to compare different dispersions

| | |
|---|---|
| $\mu = 10$ | $\mu = 100$ |
| $\sigma = 1$ | $\sigma = 10$ |
| COV = 0.1 | COV = 0.1 |

# Root Mean Square

$$RMS = \frac{\sqrt{\sum_{i=1}^{N} x_i^2}}{N}$$

The rms is equal to the standard deviation when the mean is 0

# Skewness

Skewness is a measure of the asymmetry of the data around the sample mean. If skewness is negative, the data are spread out more to the left of the mean than to the right. If skewness is positive, the data are spread out more to the right. The skewness of the normal distribution (or any perfectly symmetric distribution) is zero.

$$\text{Skewness}(X) = \frac{\sum_{i=1}^{N}(x_i - \bar{x})^3}{N\sigma^3}$$

# Kurtosis

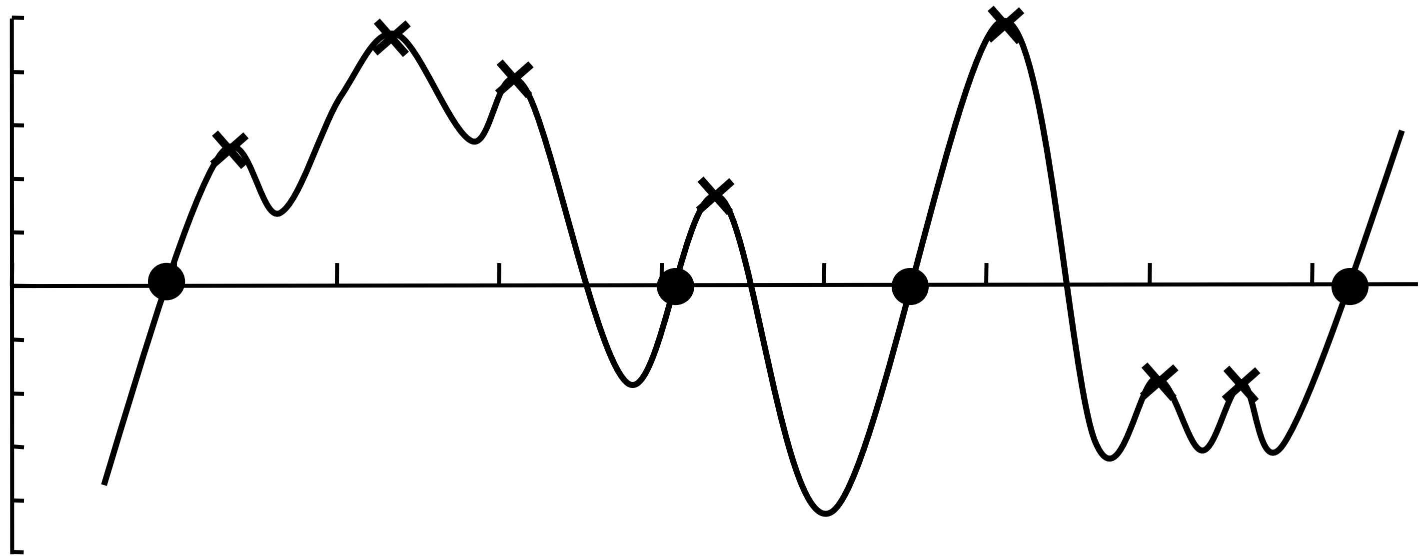
Kurtosis is a measure of how outlier-prone a distribution is. The kurtosis of the normal distribution is 3. Distributions that are more outlier-prone than the normal distribution have kurtosis greater than 3; distributions that are less outlier-prone have kurtosis less than 3.

$$\text{Kurtosis}(X) = \frac{\sum_{i=1}^{N} (x_i - \bar{x})^4}{N\sigma^4}$$

# Crest Factor

The crest factor is the ration of the peak (maximum) value to the root-mean-square (RMS) value. A sine wave has a crest factor of 1.414.

# Irregularity Factor



- ● Positive zero crossing
- × Peak

$$IF = \frac{E(0^+)}{E(P)} = \frac{4}{7}$$

IF $\rightarrow$ 1 is narrow band signal

IF $\rightarrow$ 0 is wide band signal

# Time Domain Nomenclature

- Random
- Stochastic
- Stationary
- Non-stationary
- Gaussian
- Narrow-band
- Wide-band

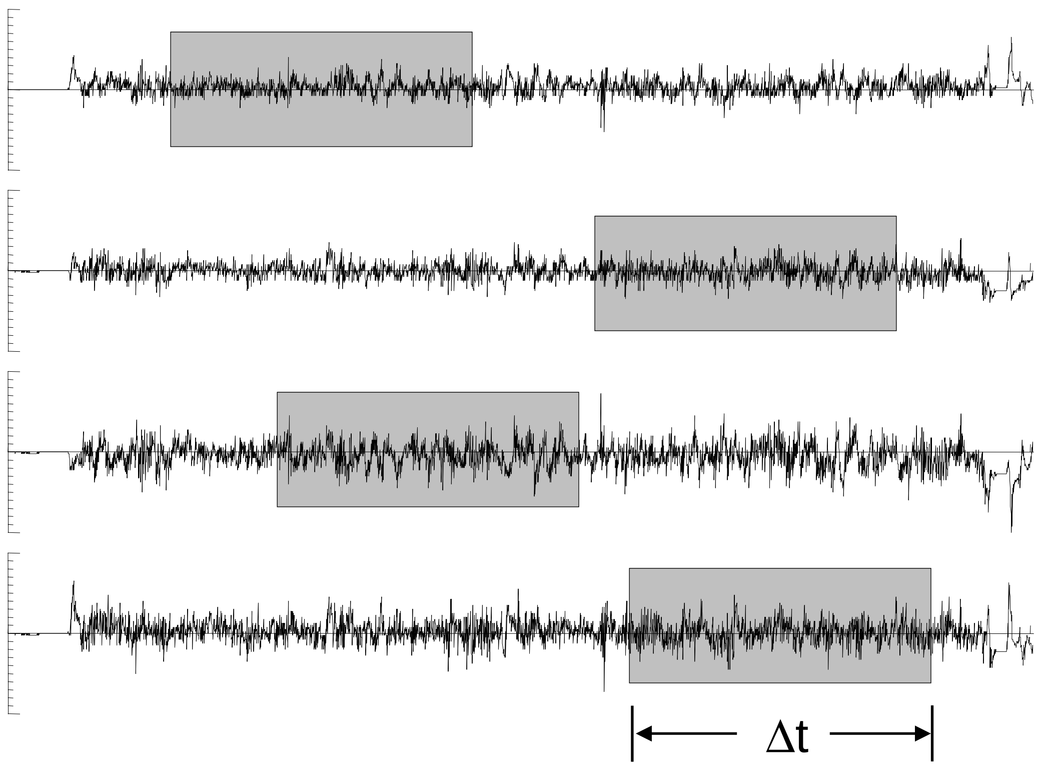
# Random

The instantaneous value can not be predicted at any future time.

# Stochastic

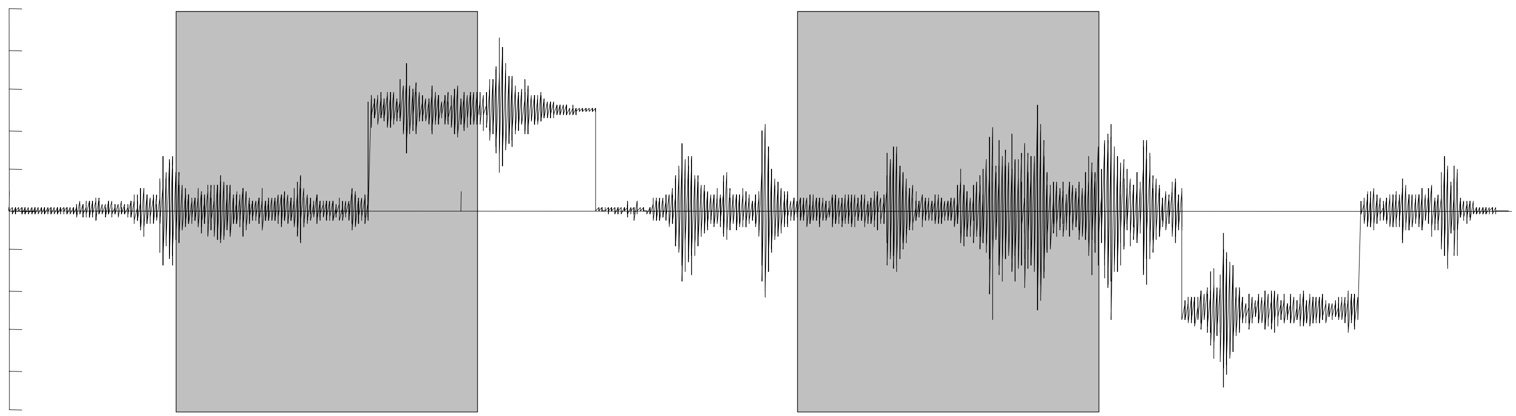
Stochastic processes provide suitable models for physical systems where the phenomena is governed by probabilities.

# Stationary



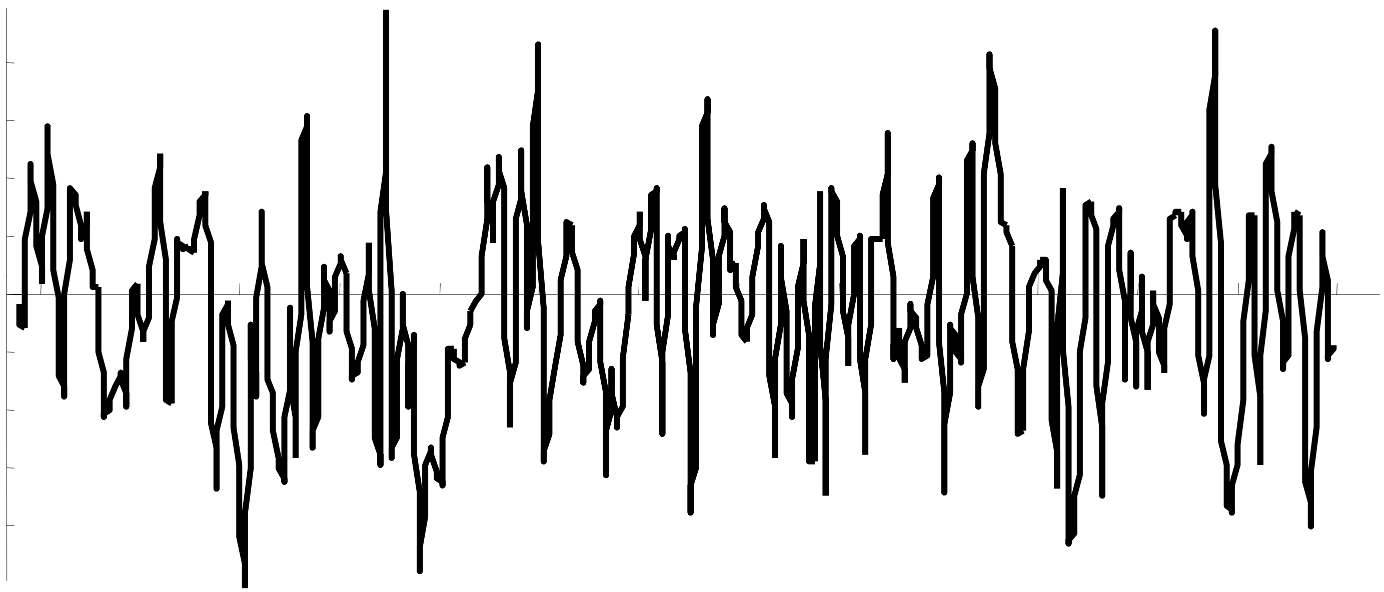

The properties computed over short time intervals, t + Δt, do not significantly vary from each other

# Non-stationary

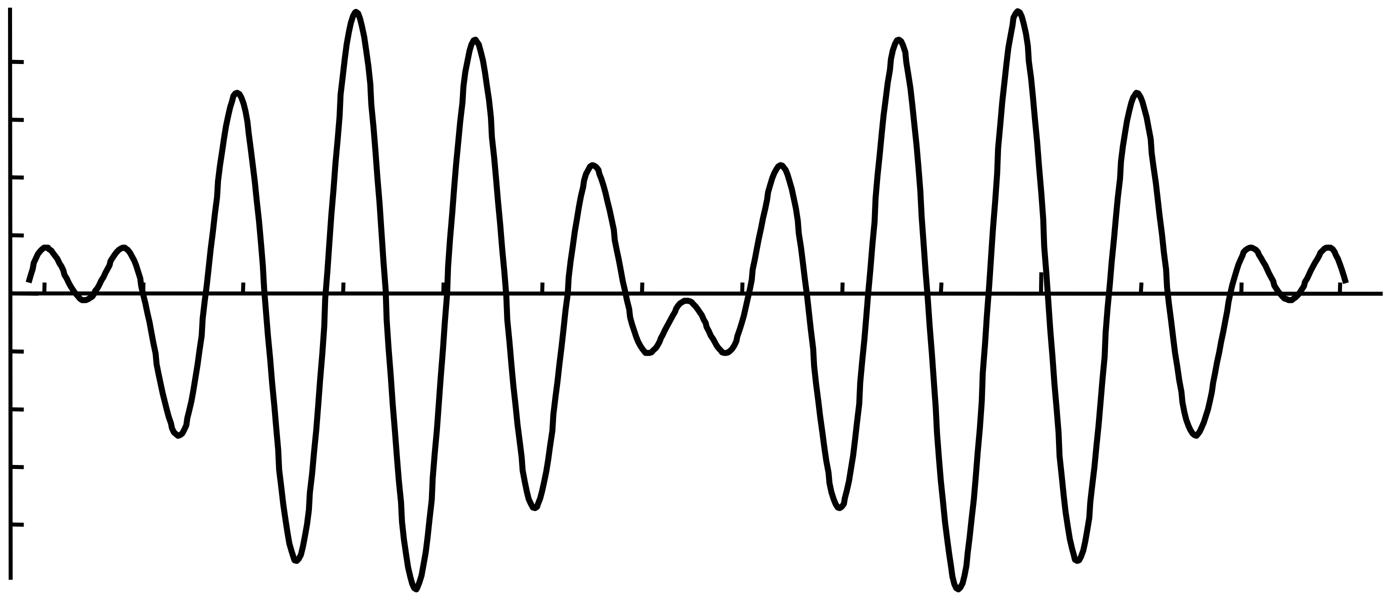


# Gaussian

Normally distributed around the mean

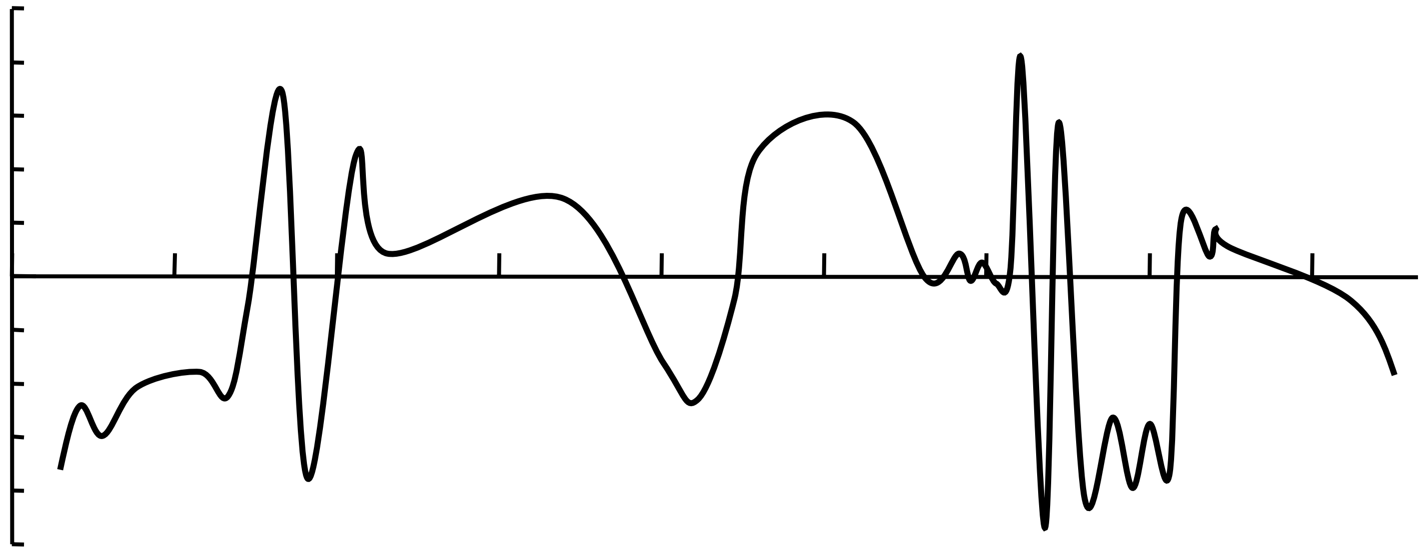


# Narrow-band



# Wide-band



# Frequency Domain Nomenclature

Fourier

Nyquist frequency

Aliasing

Anit-aliasing filter

FFT

Inverse FFT

Autospectral density

Transfer Function

# Fourier 1768 - 1830

Fourier studied the mathematical theory of heat conduction.
He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

# Fourier Series

$$X(t) = a_o + \sum_{k=1}^{\infty} a_k \sin(k\omega_o t) + b_k \cos(k\omega_o t)$$

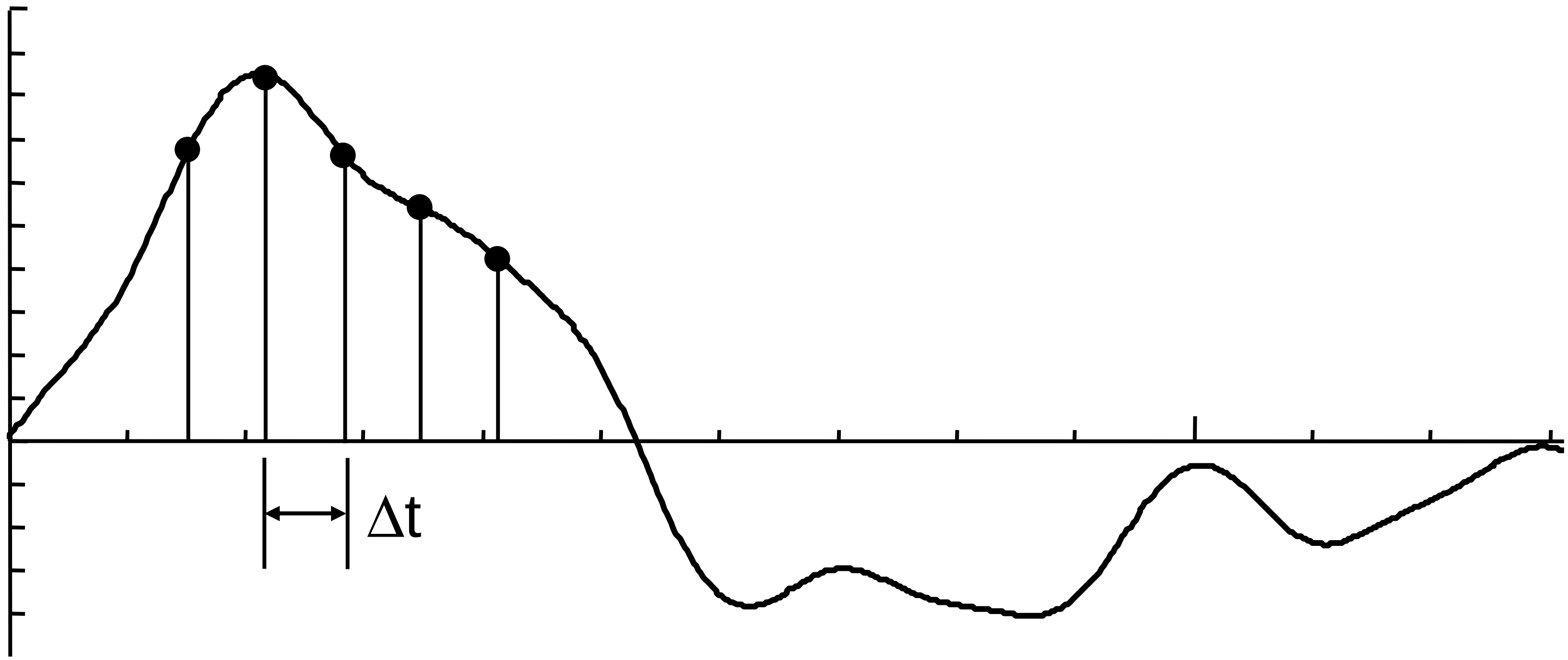
$$X(t) = c_o + \sum_{k=1}^{\infty} c_k \cos(k\omega_o t + \theta_k)$$

$$X(t) = \sum_{k=-\infty}^{\infty} X_S[k] e^{jk\omega_o t}$$

$a_k$, $b_k$, $c_k$, $X_S[k]$ are Fourier coefficients

frequency, magnitude, and phase are all described by the coefficients

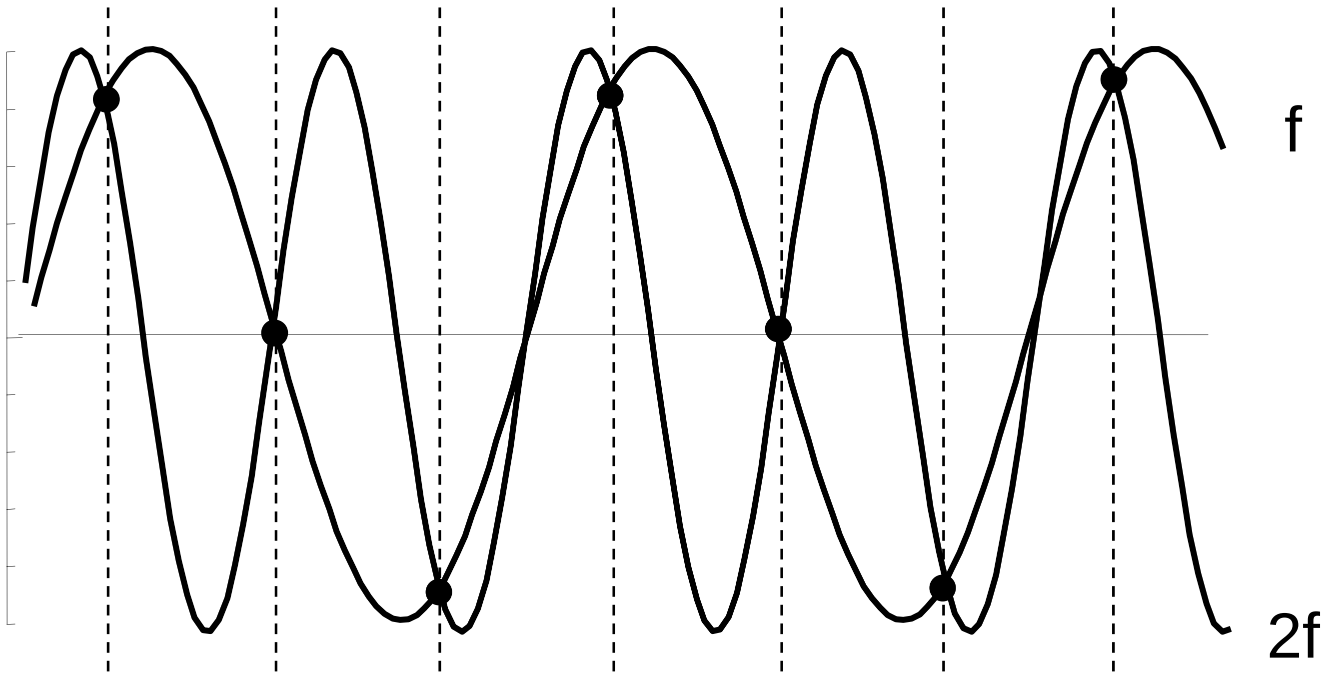
# Digital Sampling




What Δt should be used for sampling the data?

# Aliasing

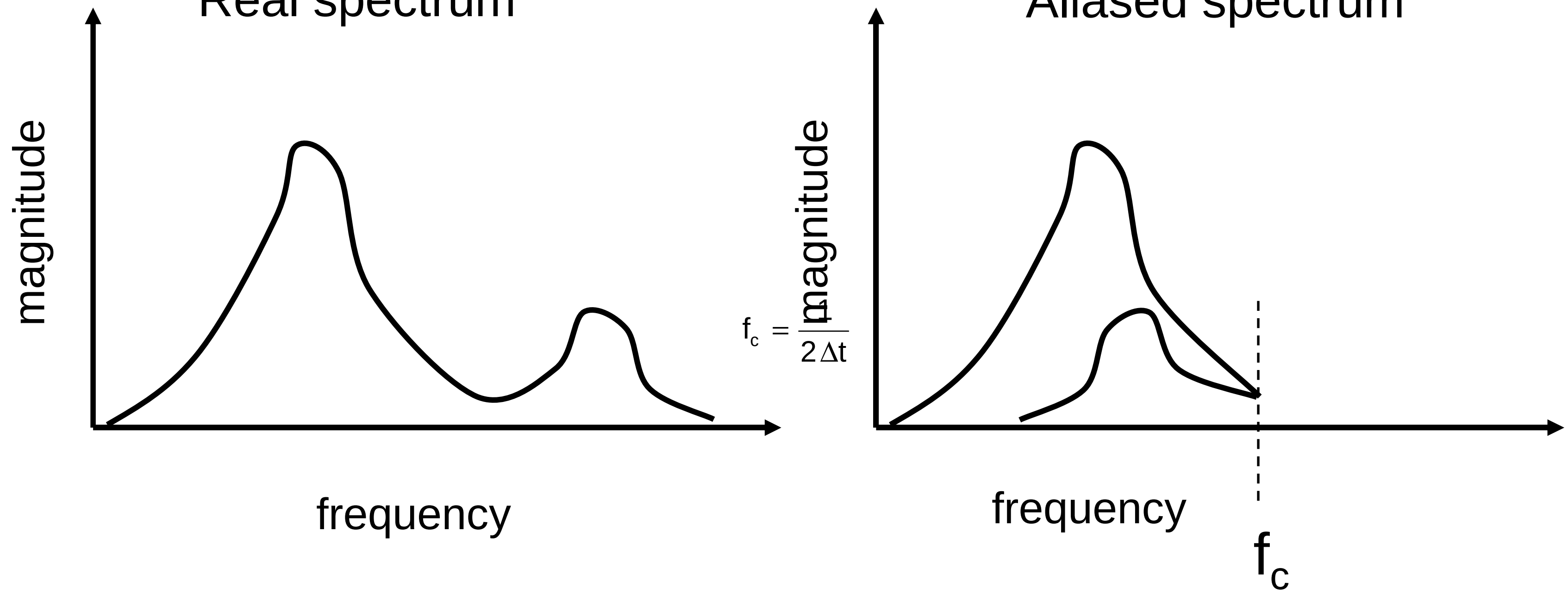
2 signals that differ in frequency by a factor of 2




Both signals have the same digital samples.

# Nyquist frequency

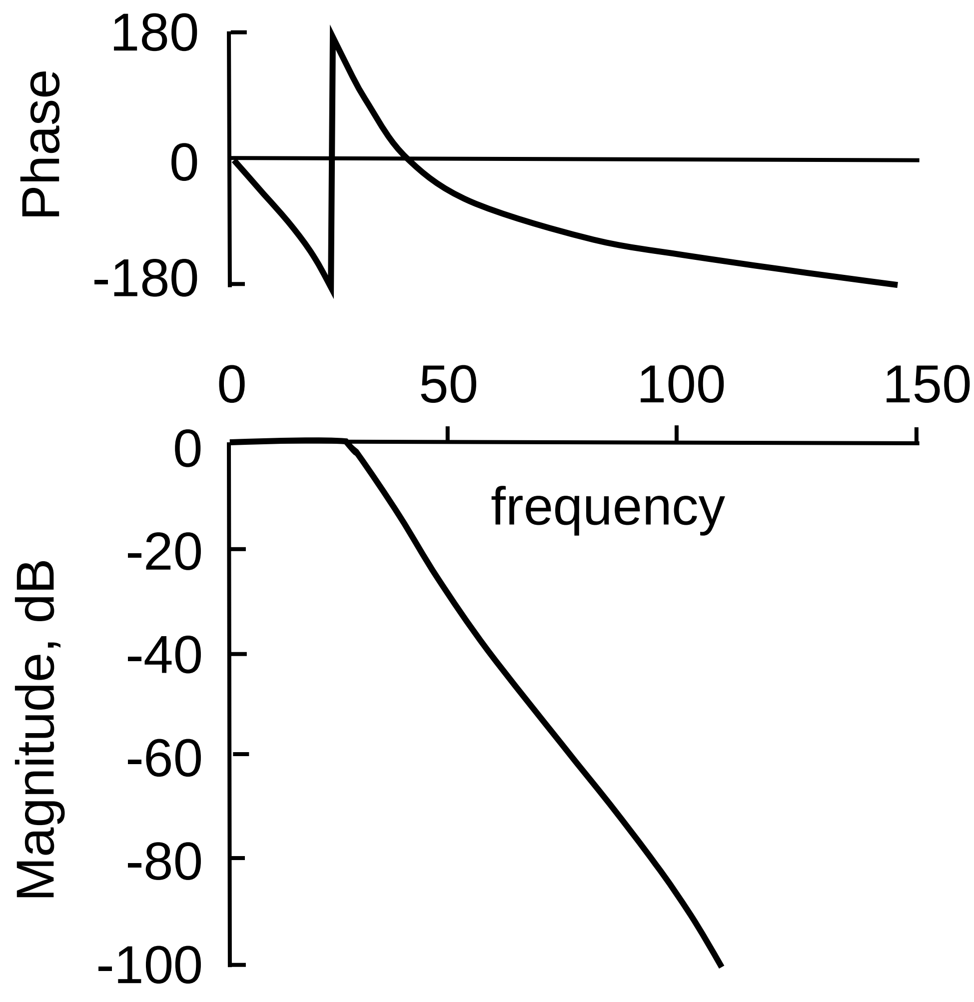
Real spectrum — Aliased spectrum



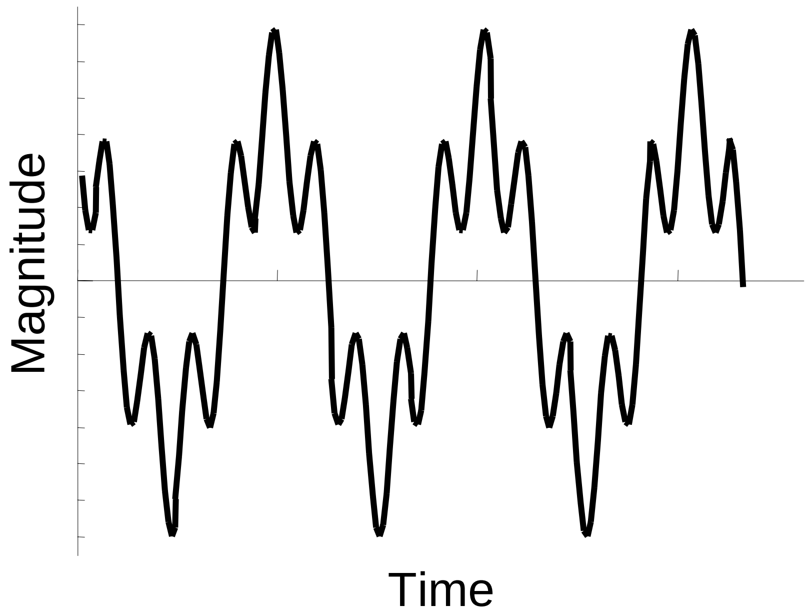

$$f_c = \frac{1}{2\Delta t}$$

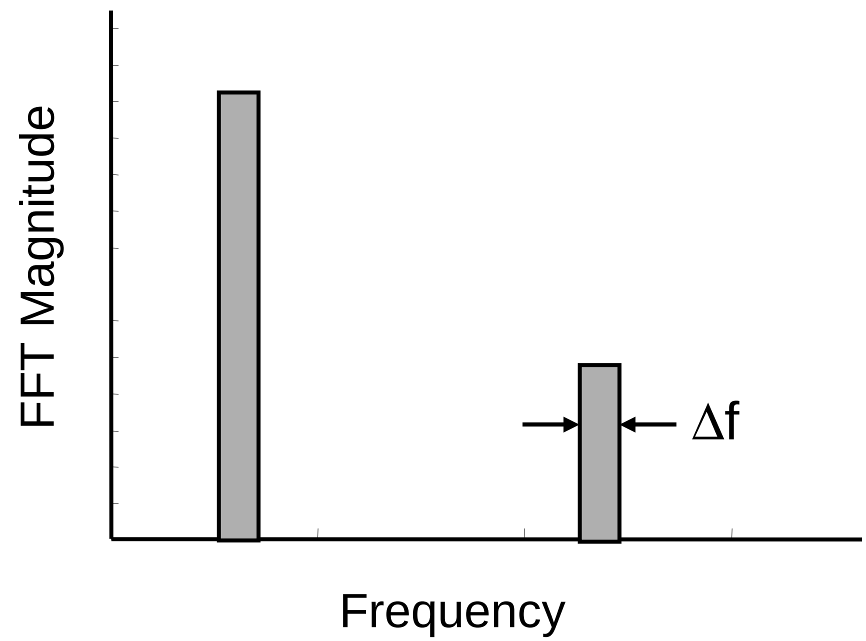
Sample at 2 times the frequency of interest

# Anti-aliasing Filter



# Fourier Transform

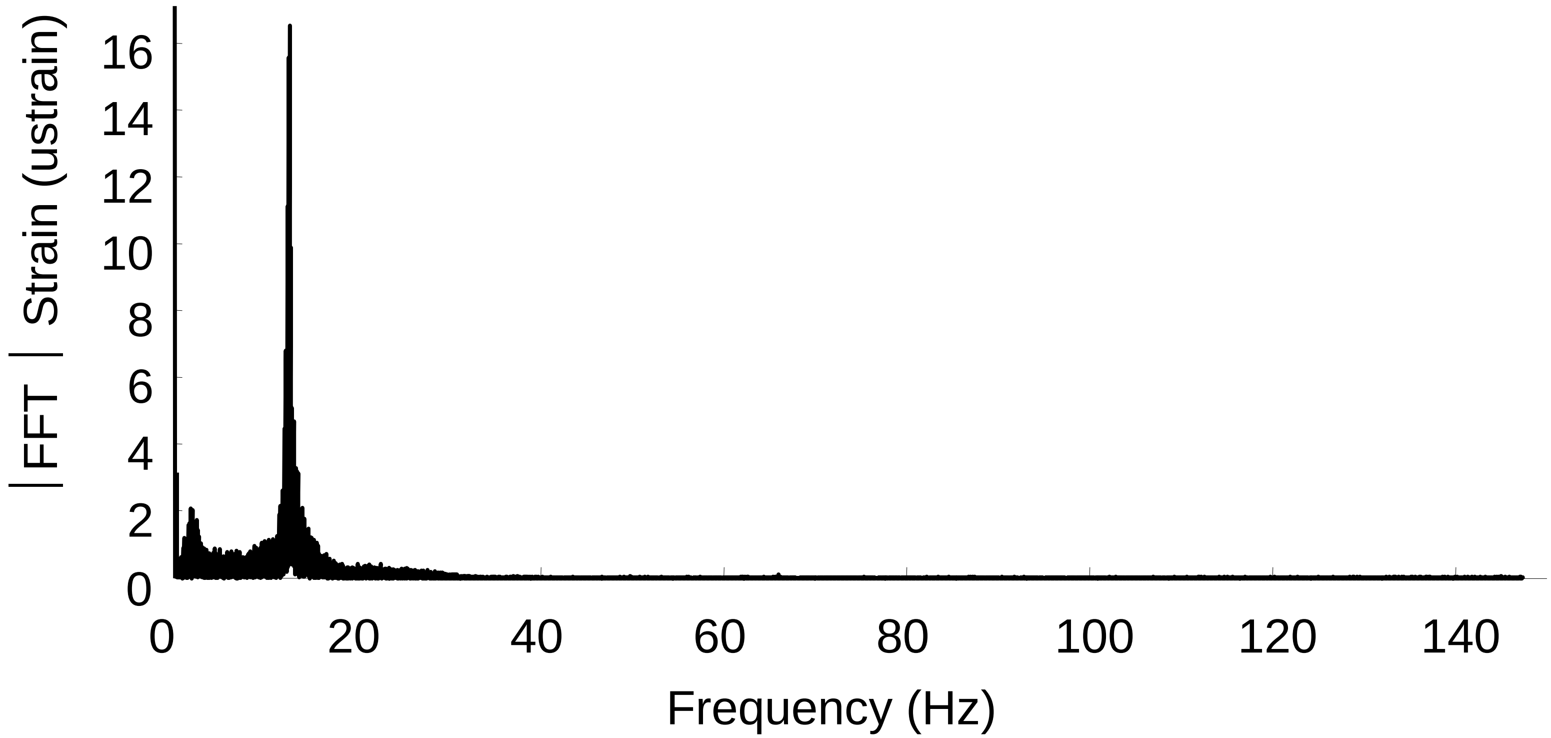



The area under each spike represents the magnitude of the sine wave at that frequency.

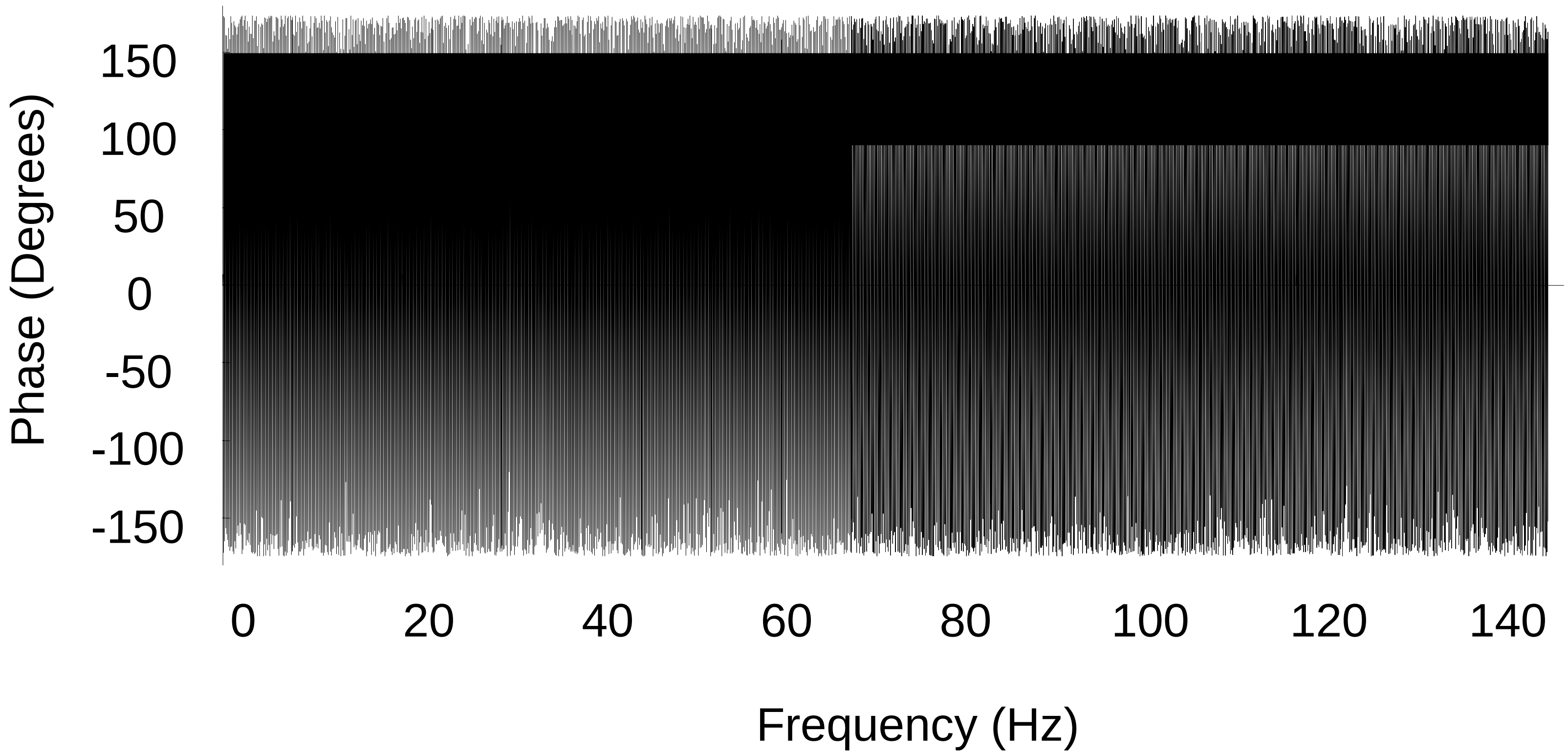
The magnitude of the FFT depends on the frequency window Δf.

# Fast Fourier Transform - FFT




$$X(t) = c_o + \sum_{k=1}^{N/2} c_k \cos(k\omega_o t + \theta_k)$$

# Fast Fourier Transform - FFT

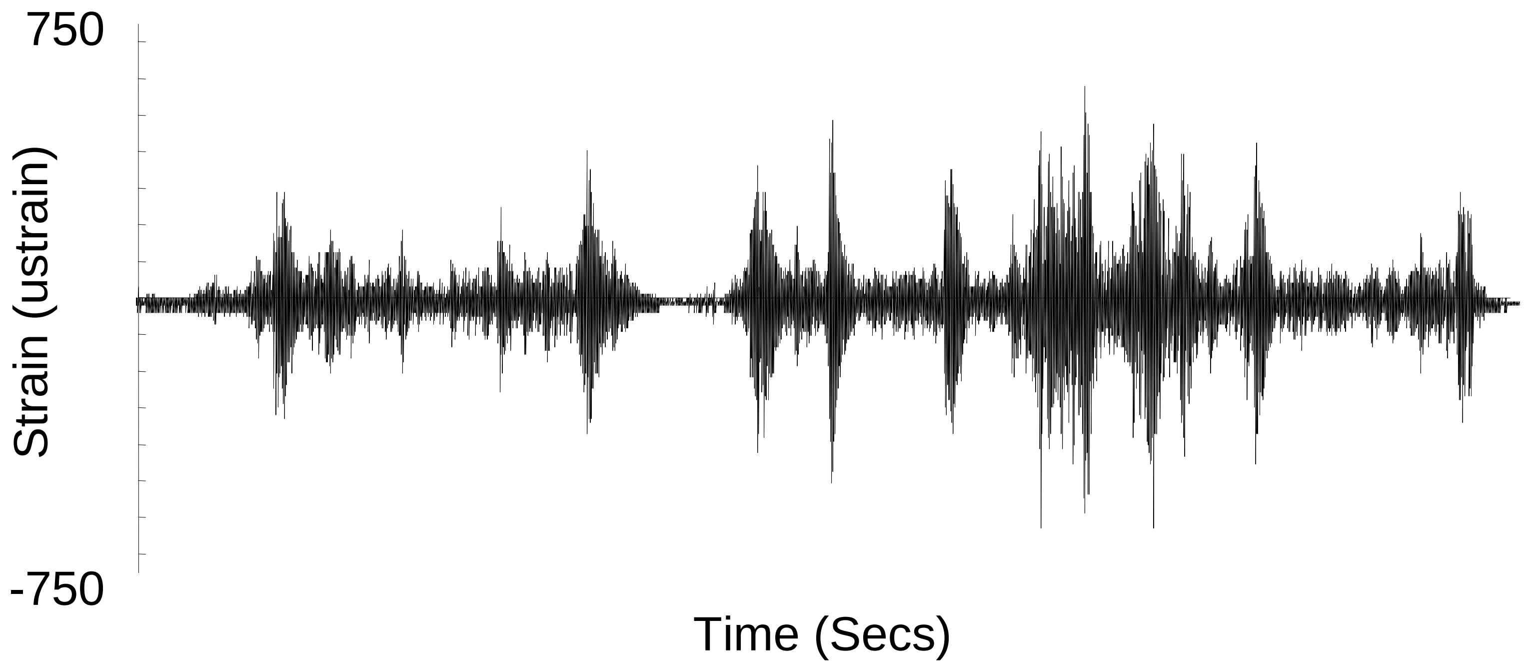



$$X(t) = c_o + \sum_{k=1}^{N/2} c_k \cos(k\omega_o t + \theta_k)$$

# Inverse FFT

$$X(t) = c_o + \sum_{k=1}^{N/2} c_k \cos(k\omega_o t + \theta_k)$$

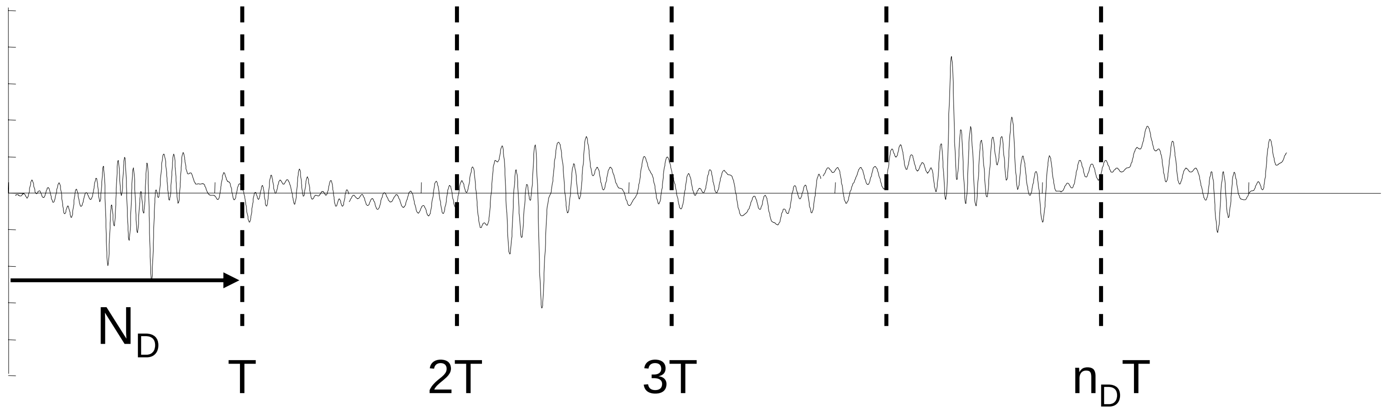
$c_k$, $\theta_k$, and $k\omega_o$ are all known



# Autospectral Density

| Function | Units | |
|---|---|---|
| Time History | EU | $X(n)$ |
| Linear Spectrum | EU | $S(n) = DFT(X(n))$ |
| AutoPower | EU^2 | $AP(n) = S(n) \cdot S(n)$ |
| PSD | (EU^2)/Hz | $PSD(n) = AP(n) / ( W_f \cdot \Delta f )$ |
| ESD | (EU^2*sec)/Hz | $ESD(n) = AP(n) \cdot T / ( W_f \cdot \Delta f )$ |

# Calculating Autospectral Density

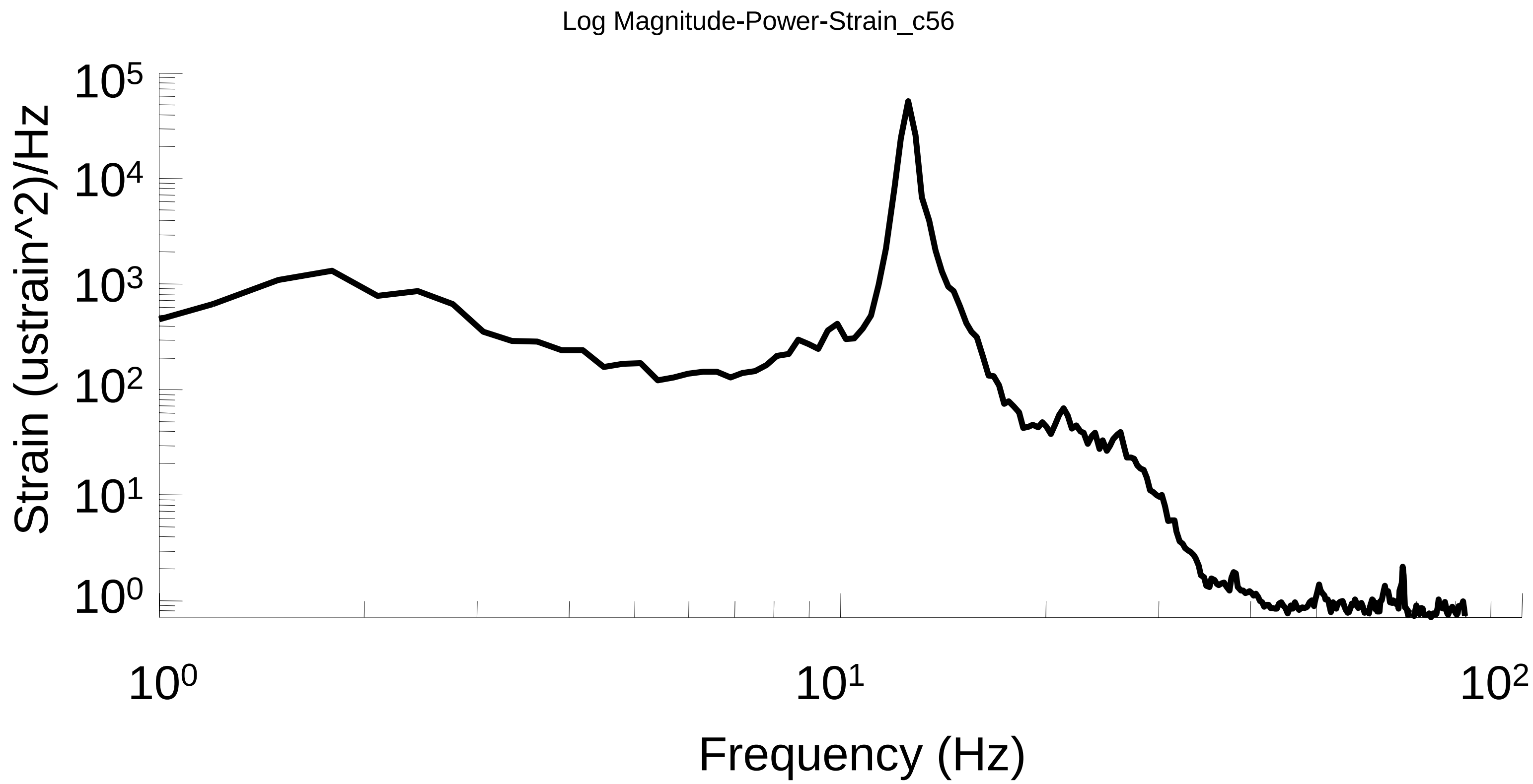



$N_D$ Block size

$$S(f_k) = \frac{1}{n_D N_D \Delta t} \sum_{i=1}^{n_D} \left| X(f_k) \right|^2$$

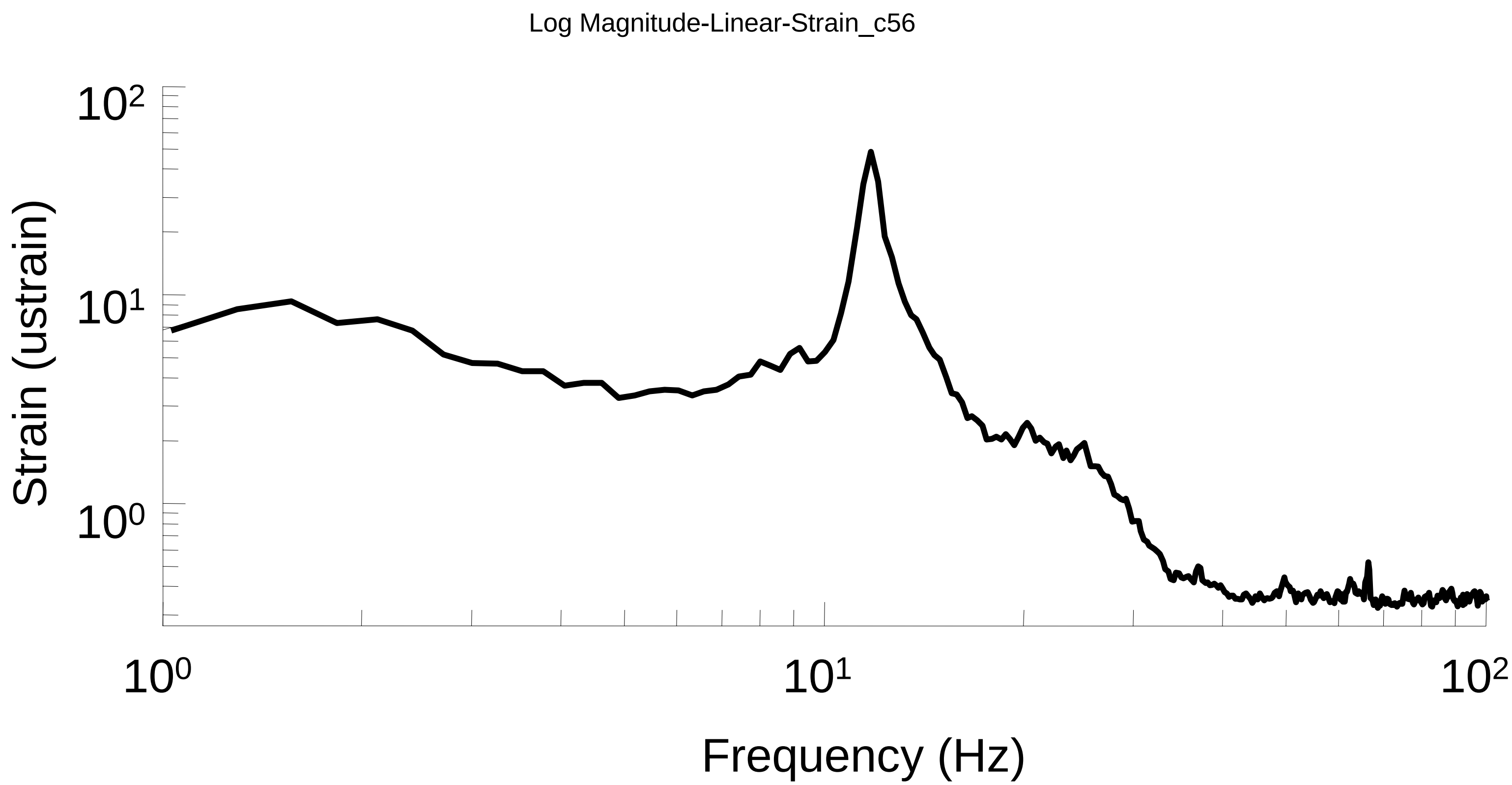
Magnitude only no phase information

# Power Spectral Density - PSD



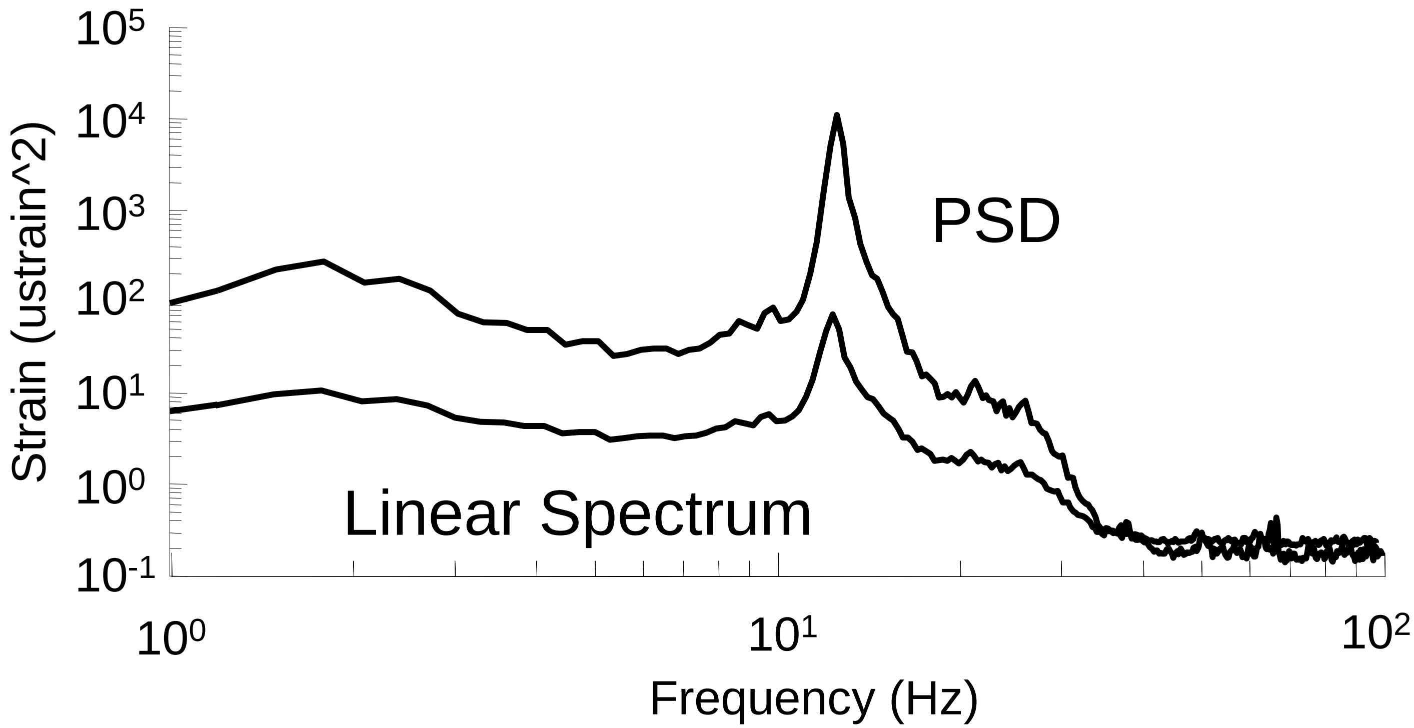

Average power associated with a 1 Hz frequency window centered at each frequency, f .
Phase information is lost.

# Linear Spectrum




Sometimes called Amplitude Spectral Density

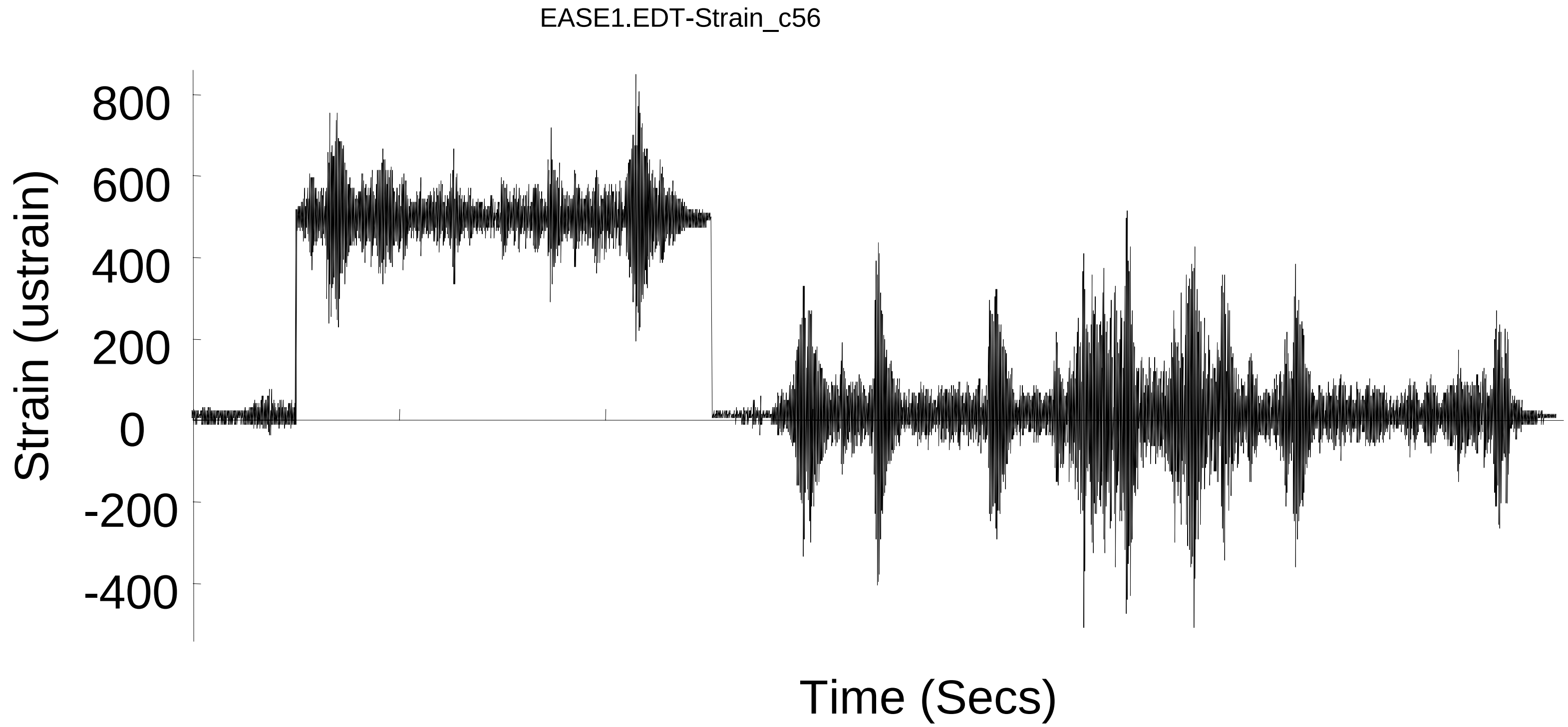
# Comparison



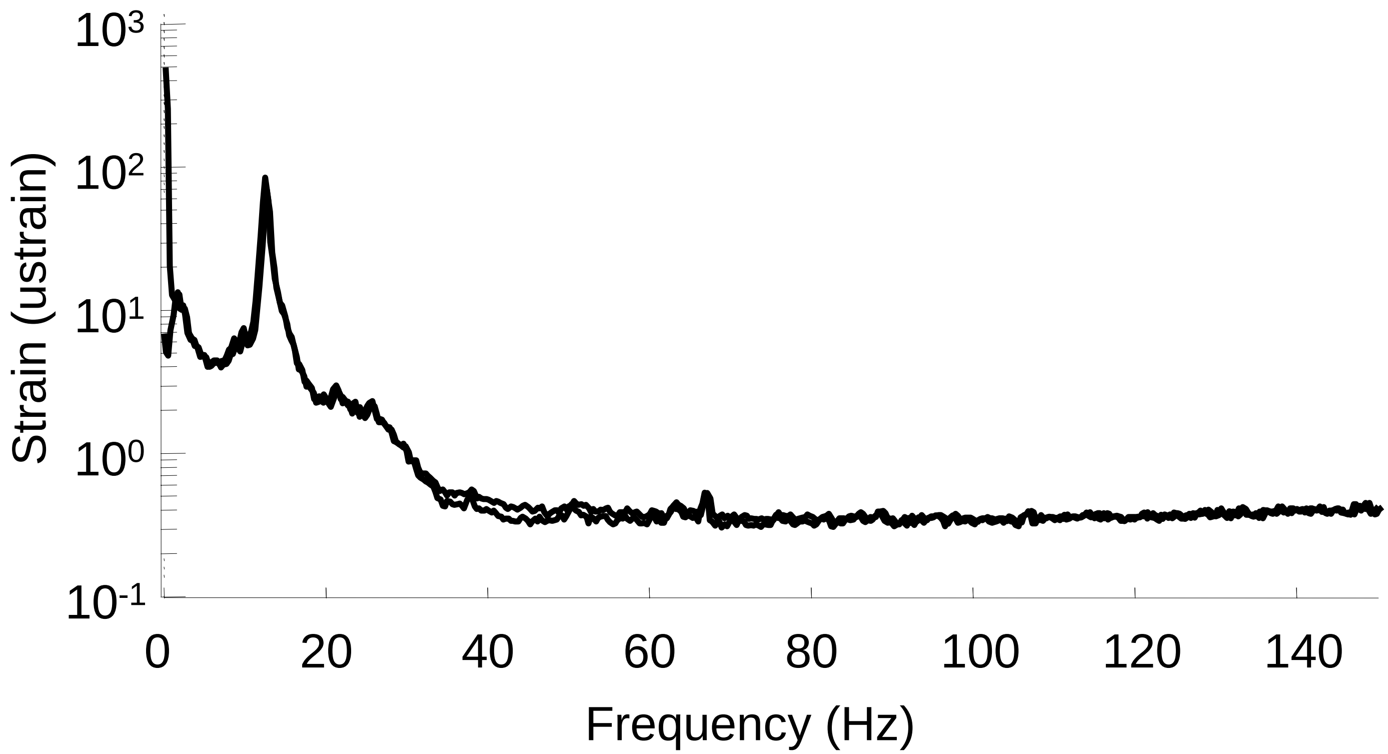
$10^5$
$10^4$
$10^3$
$10^2$
$10^1$
$10^0$
$10^{-1}$
Strain (ustrain^2)
PSD
Linear Spectrum
$10^0$
$10^1$
$10^2$
Frequency (Hz)

# Frequency Domain Limitations

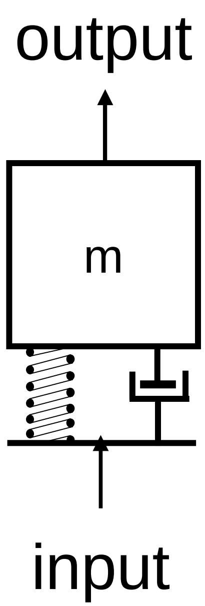
Non-stationary signals

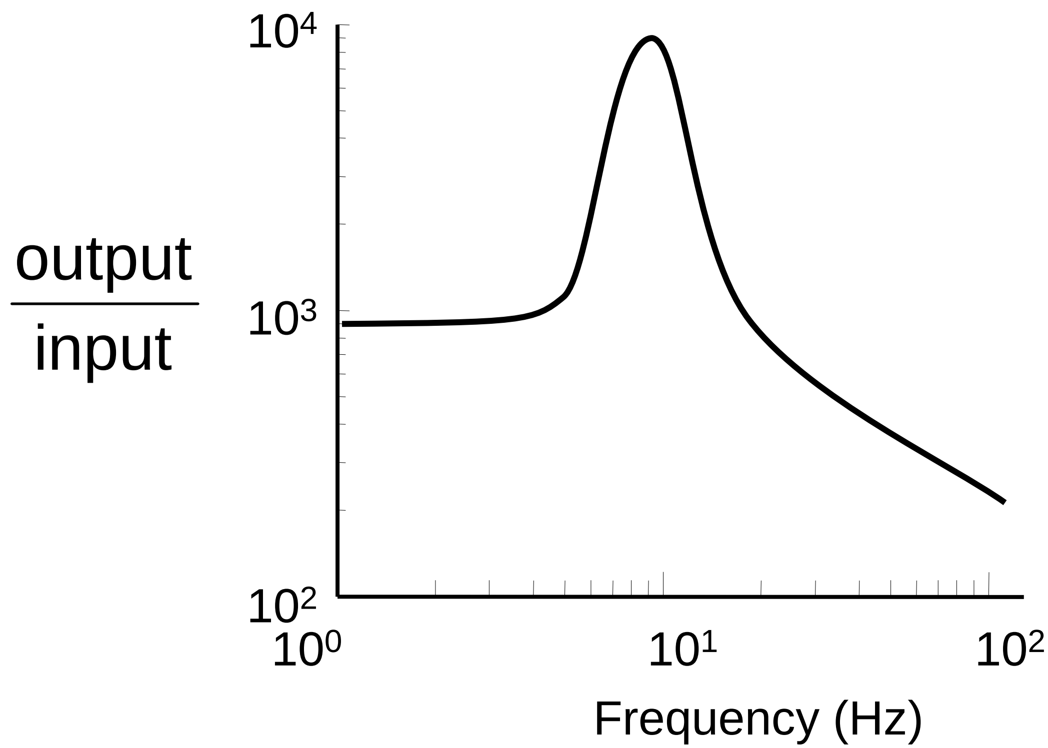


# Linear Spectrum



# Transfer Function



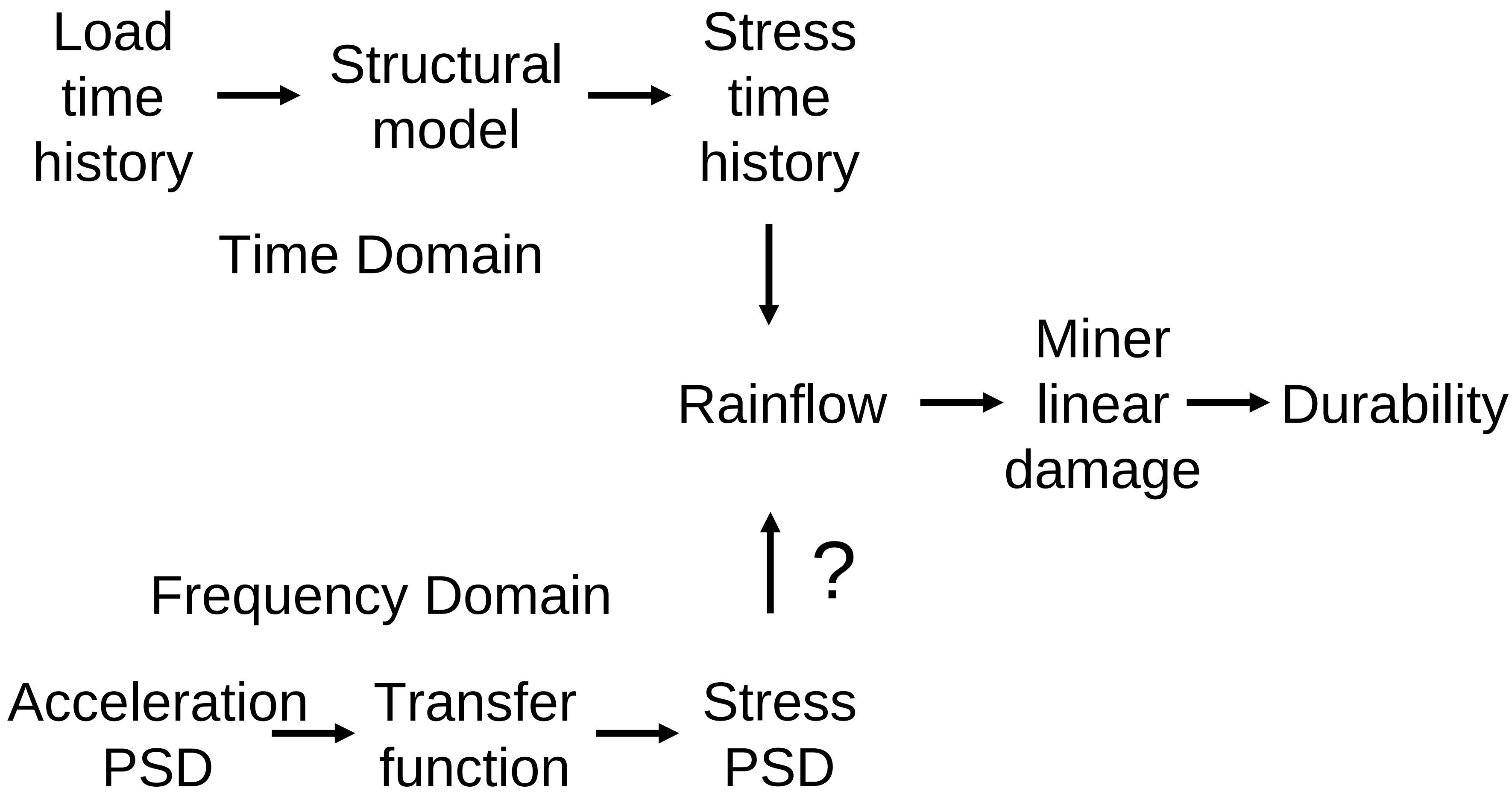
# Frequency Based Fatigue

- Stationary Loading
  - Wind
  - Sea State
  - Vibration

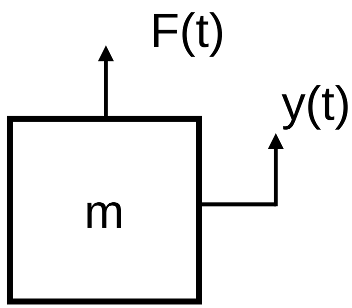
# Assumptions

Random
Gaussian
Stationary

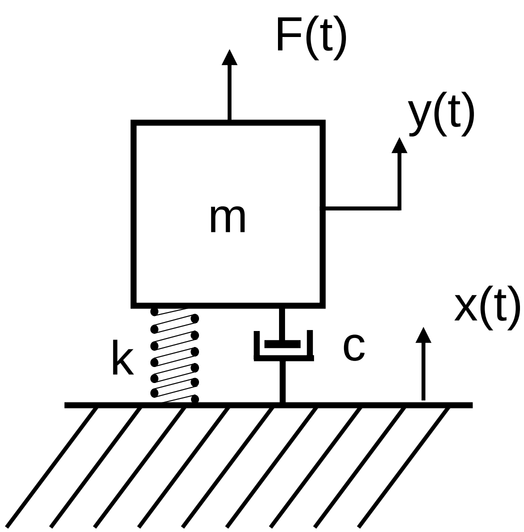
# Fatigue Analysis



# Dynamics Model




$$m\frac{d^2y}{dt^2} = F(t)$$

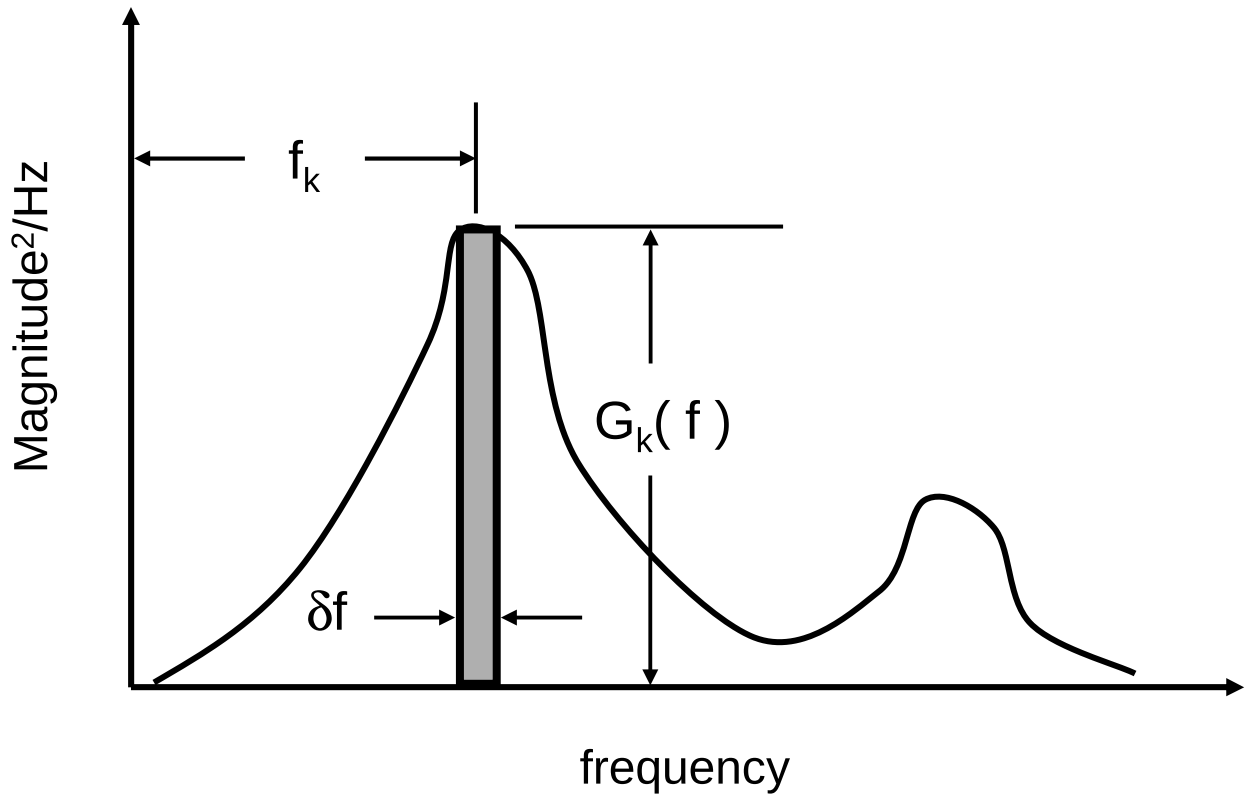



$$z(t) = y(t) - x(t)$$

$$m\frac{d^2z}{dt^2} + c\frac{dz}{dt} + kz = F(t) - m\frac{d^2x}{dt^2}$$

$$\frac{d^2z}{dt^2} + 2\zeta\omega_n\frac{dz}{dt} + \omega_n^2 z = -\frac{d^2x}{dt^2}$$

# PSD Moments




$$m_n = \sum_{k=1}^{N} f_k^{\,n}\, G_k(f_k)\,\delta f$$

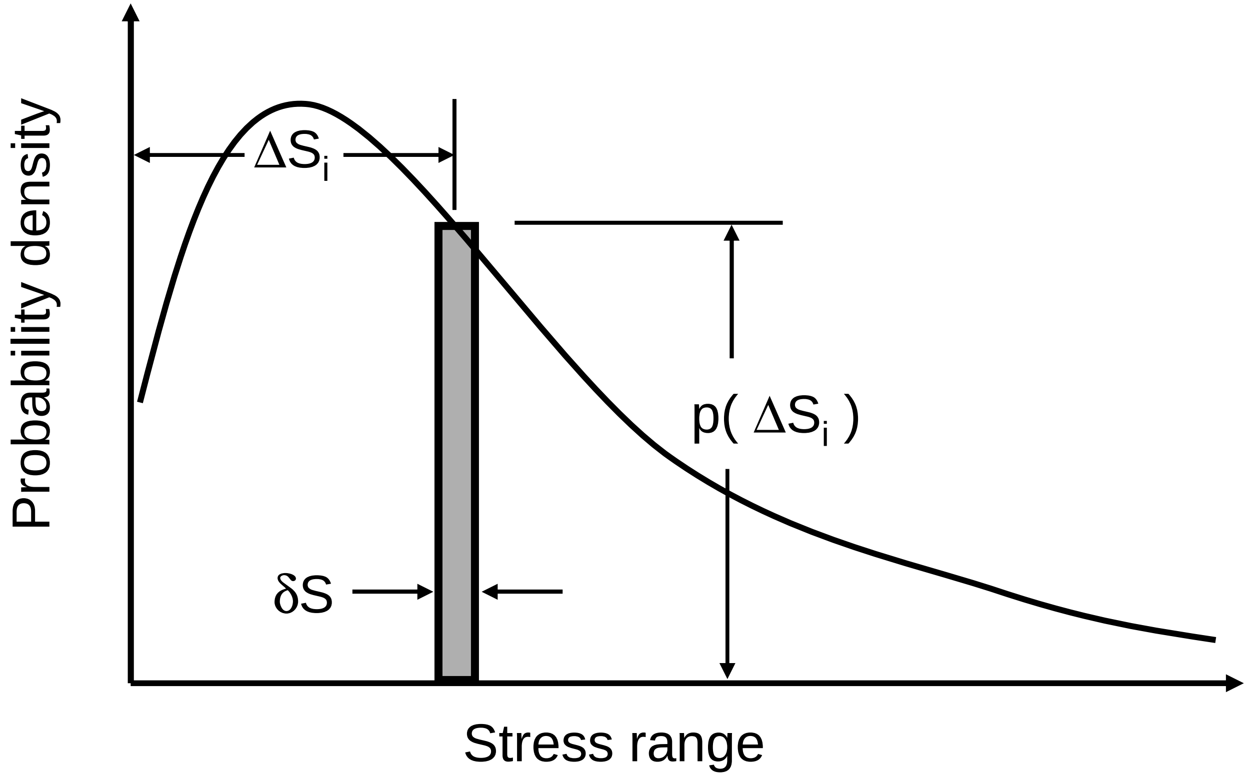
# Expected Values

zero crossings $$E(0) = \sqrt{\frac{m_2}{m_0}}$$

peaks $$E(P) = \sqrt{\frac{m_4}{m_2}}$$

Irregularity factor $$IF = \frac{E(0)}{E(P)} = \sqrt{\frac{{m_2}^2}{m_0\, m_4}}$$

# Probability Density Function




The probability $P(\Delta S_i)$ of a stress range occurring between

$$\Delta S_i - \frac{\delta S}{2} \text{ and } \Delta S_i + \frac{\delta S}{2} \text{ is } P(\Delta S_i) = p(\Delta S_i)\,\delta S$$

# Fatigue Damage

Cycles at level i $n_i = p(\Delta S_i)\,\delta S\, N_T$

Total cycles $N_T = E(P)\,T$

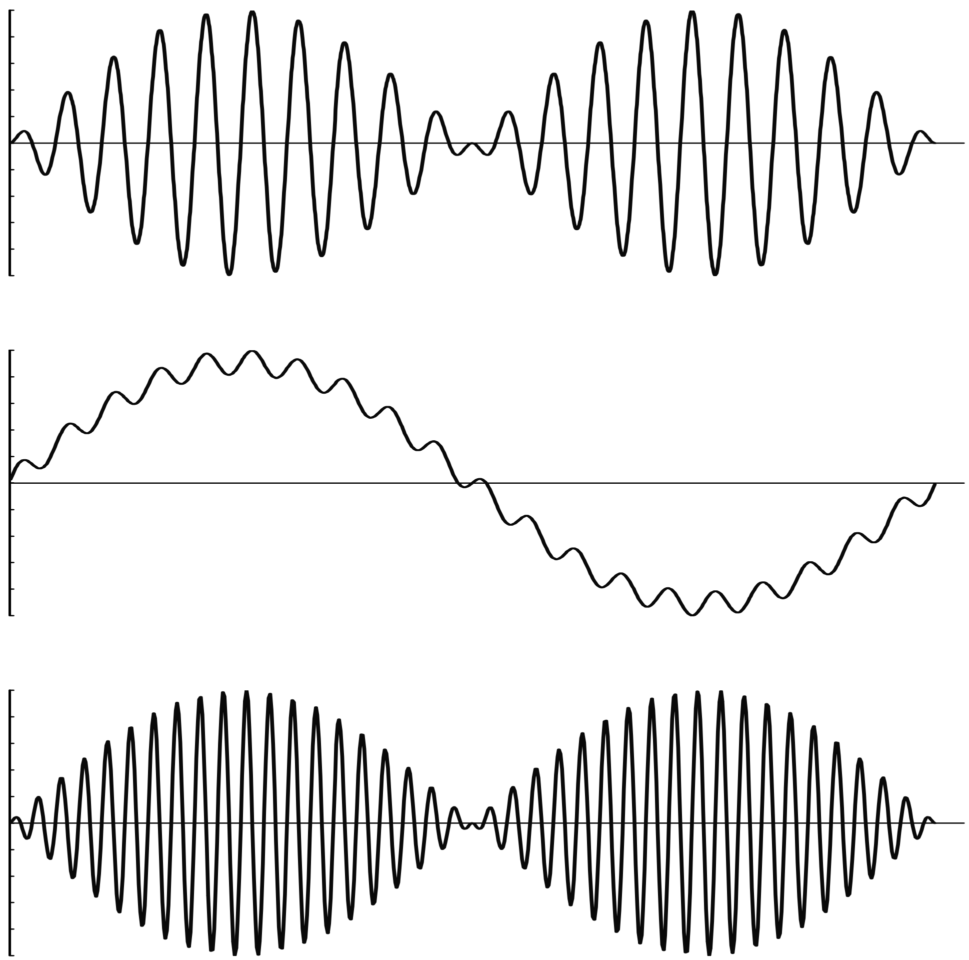
Total time

Fatigue life $$N_f(\Delta S_i) = \left(\frac{\Delta S_i}{2S_f'}\right)^{\frac{1}{b}}$$

Fatigue Damage $$D = \sum_{i=1}^{N} \frac{n_i}{N_f(\Delta S_i)} = E(P)\,T\,\delta S\,(2S_f')^{\frac{1}{b}} \sum_{i=1}^{N} \frac{p(\Delta S_i)}{(\Delta S_i)^{\frac{1}{b}}}$$

Fatigue damage is determined by $p(\Delta S_i)$

# Narrow Band Solution



Rayleigh distribution

$$p(\Delta S) = \frac{\Delta S}{4m_0}\exp\left[-\frac{\Delta S^2}{8m_0}\right]$$

IF $\rightarrow$ 1 is narrow band signal

IF $\rightarrow$ 0 is wide band signal

This wide band signal has the same peak distribution as this narrow band signal

# Dirlik Solution

$$p(\Delta S) = f(m_0, m_1, m_2, m_4)$$

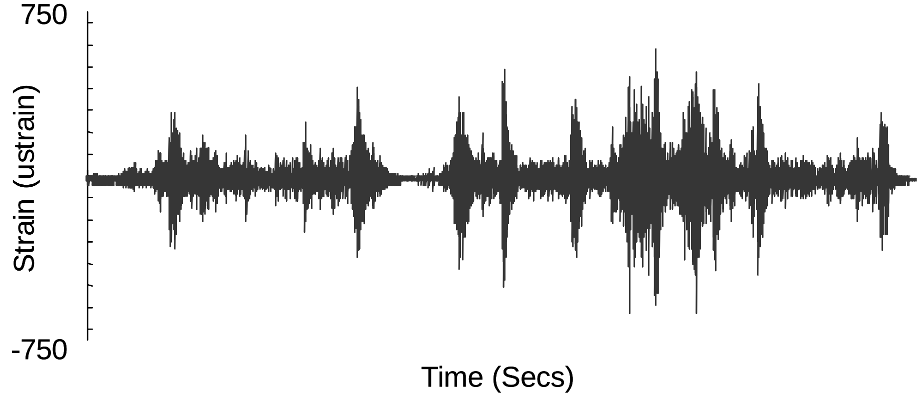
$$p(\Delta S)=\frac{\frac{D_1}{Q}\exp\left(-\frac{Z}{Q}\right)+\frac{D_2}{R^2}\exp\left(-\frac{Z^2}{R^2}\right)+D_3\,Z\exp\left(-\frac{Z^2}{2}\right)}{2\sqrt{m_0}}$$

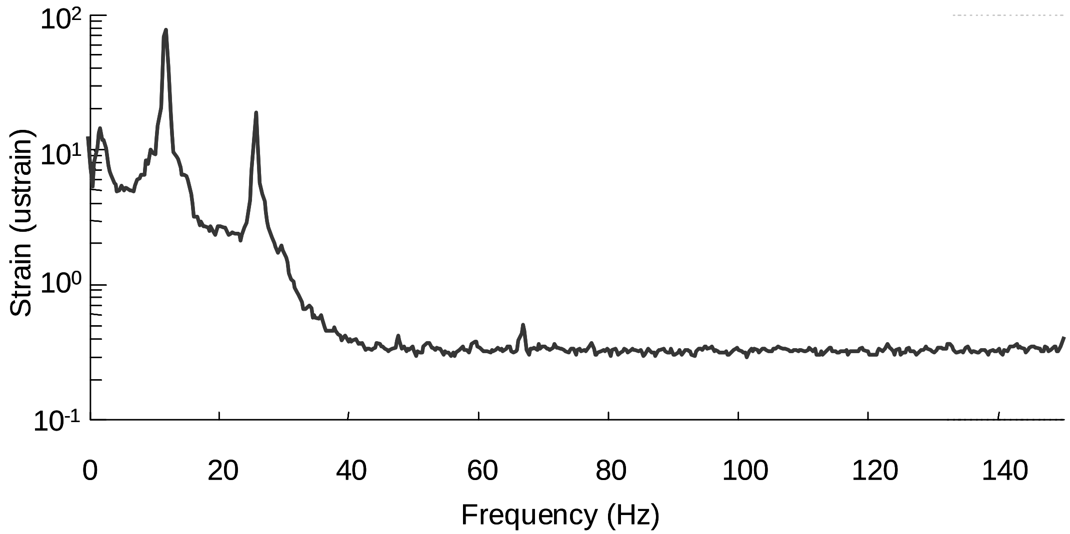
$$Z=\frac{\Delta S}{2\sqrt{m_0}} \qquad IF=\frac{m_2}{\sqrt{m_0\,m_4}} \qquad X_m=\frac{m_1}{m_0}\sqrt{\frac{m_2}{m_4}} \qquad D_1=\frac{2\left(X_m-IF^2\right)}{1+IF^2}$$

$$R=\frac{IF-X_m-D_1^{\,2}}{1-IF-D_1+D_1^{\,2}} \qquad D_2=\frac{1-IF-D_1+D_1^{\,2}}{1-R} \qquad Q=\frac{5\left(IF-D_1-D_2\,R\right)}{4D_1}$$

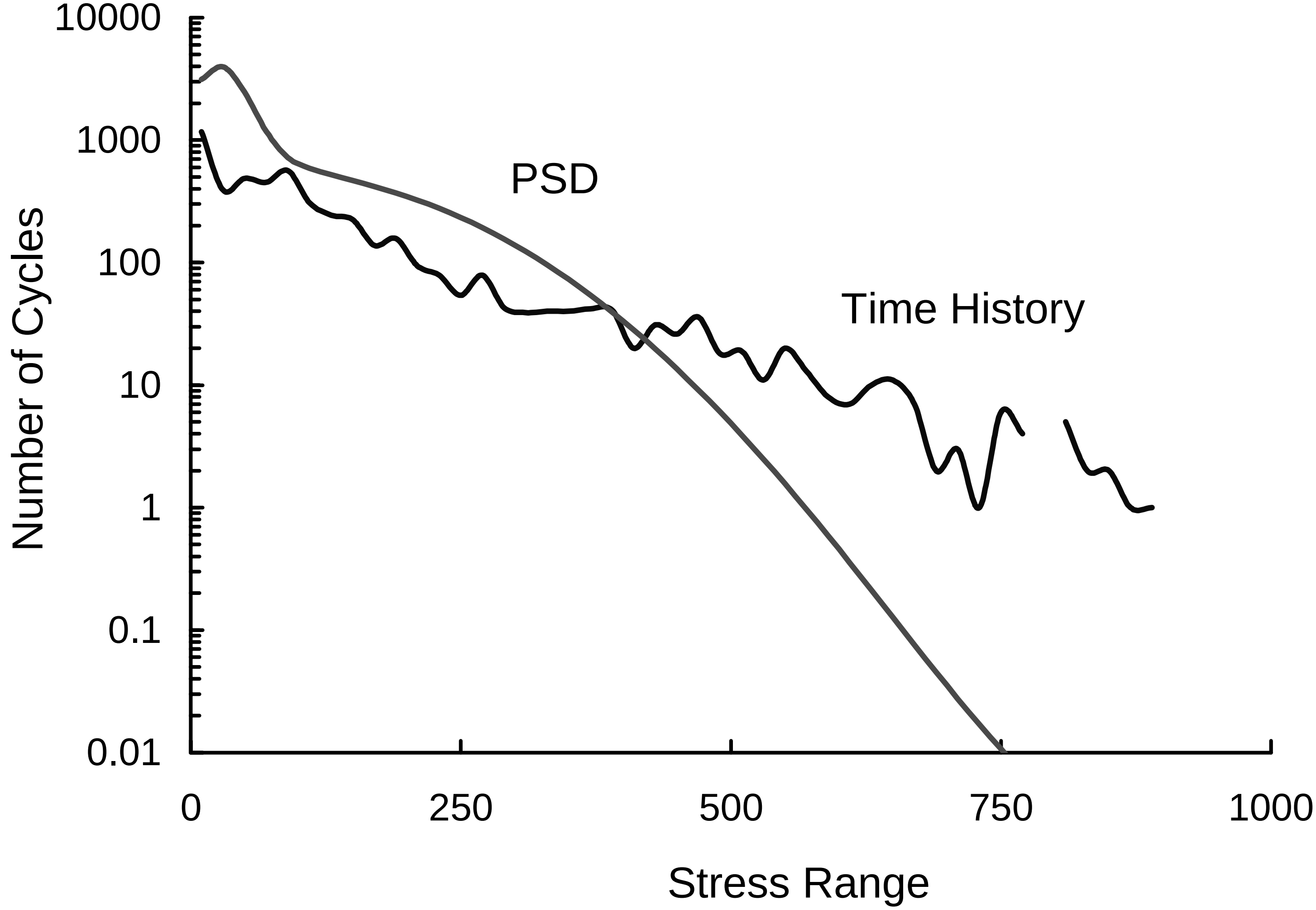
$$D_3=1-D_1-D_2$$

# Loading History



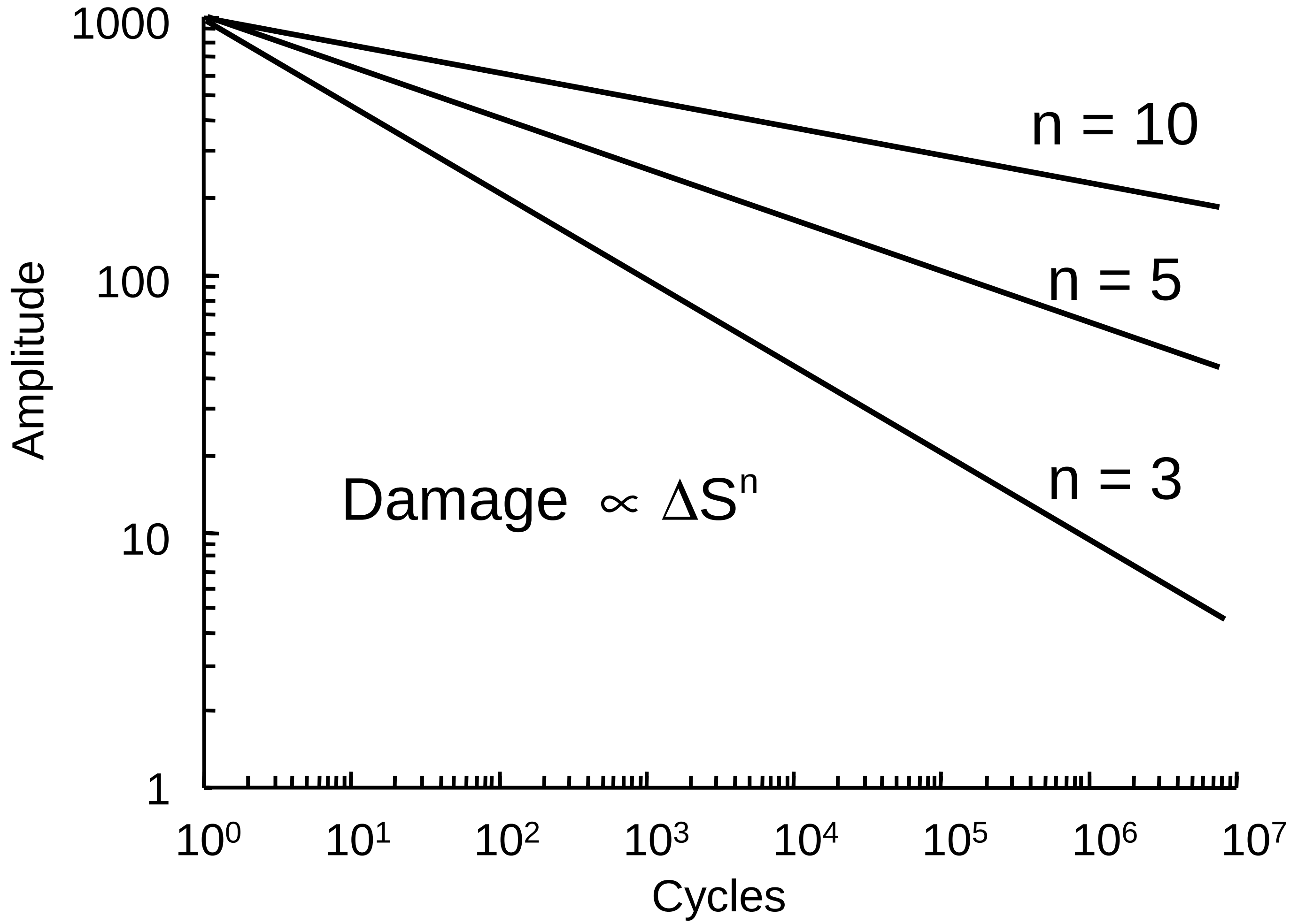
# Rainflow Ranges



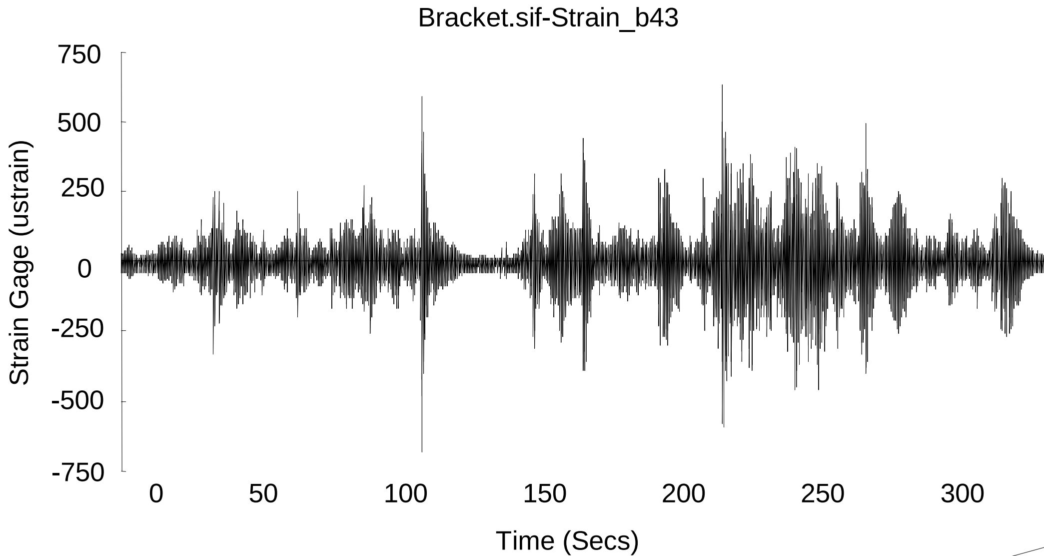
# Relative Fatigue Estimates

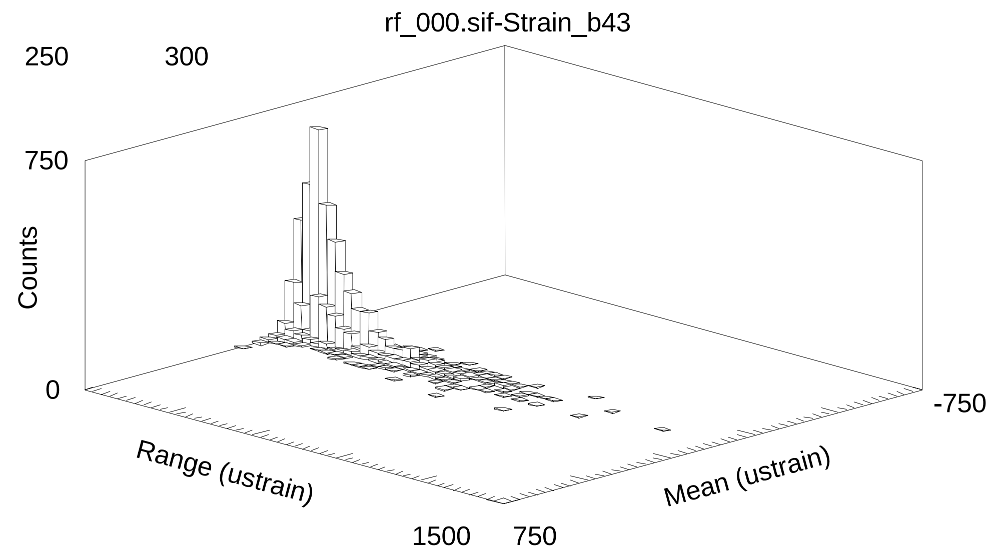
| SN Slope | Dirlik $\frac{N_{PSD}}{N_{TH}}$ | Rayleigh $\frac{N_{PSD}}{N_{TH}}$ |
|---|---|---|
| 10 | 371 | 109 |
| 5 | 6.6 | 1.9 |
| 3 | 1.8 | 0.5 |

# Fatigue Data



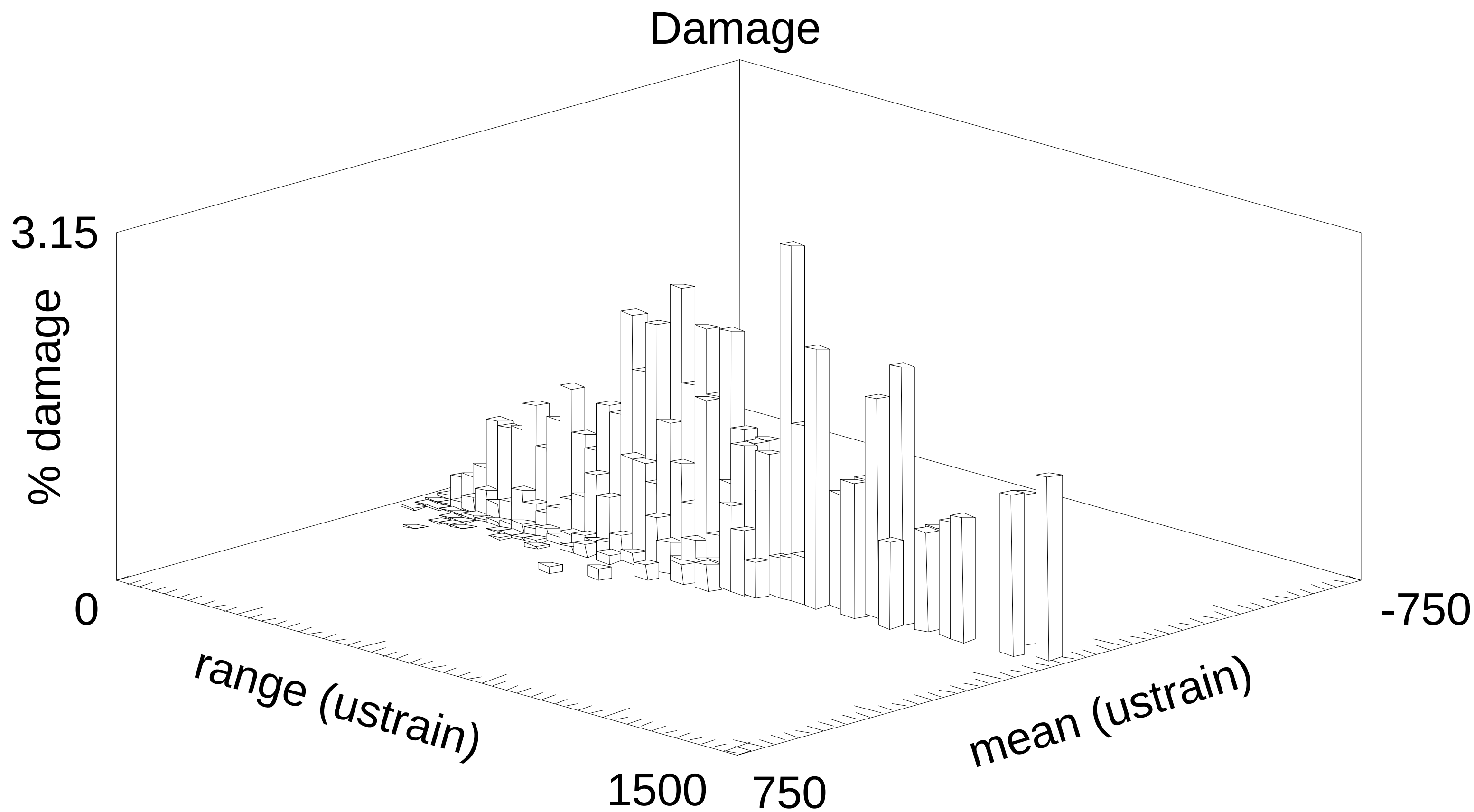
# Loading History



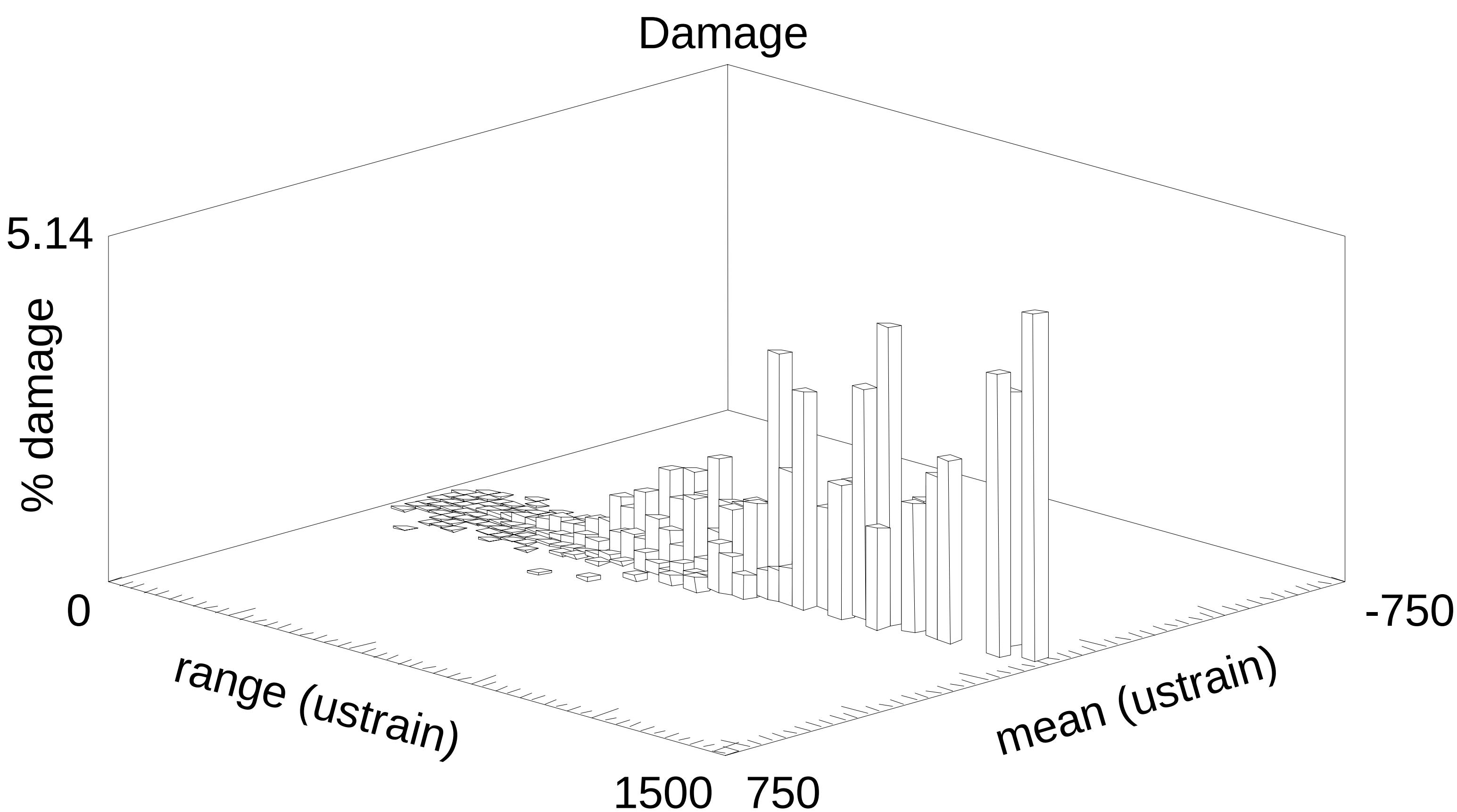

  

# Slope = 3

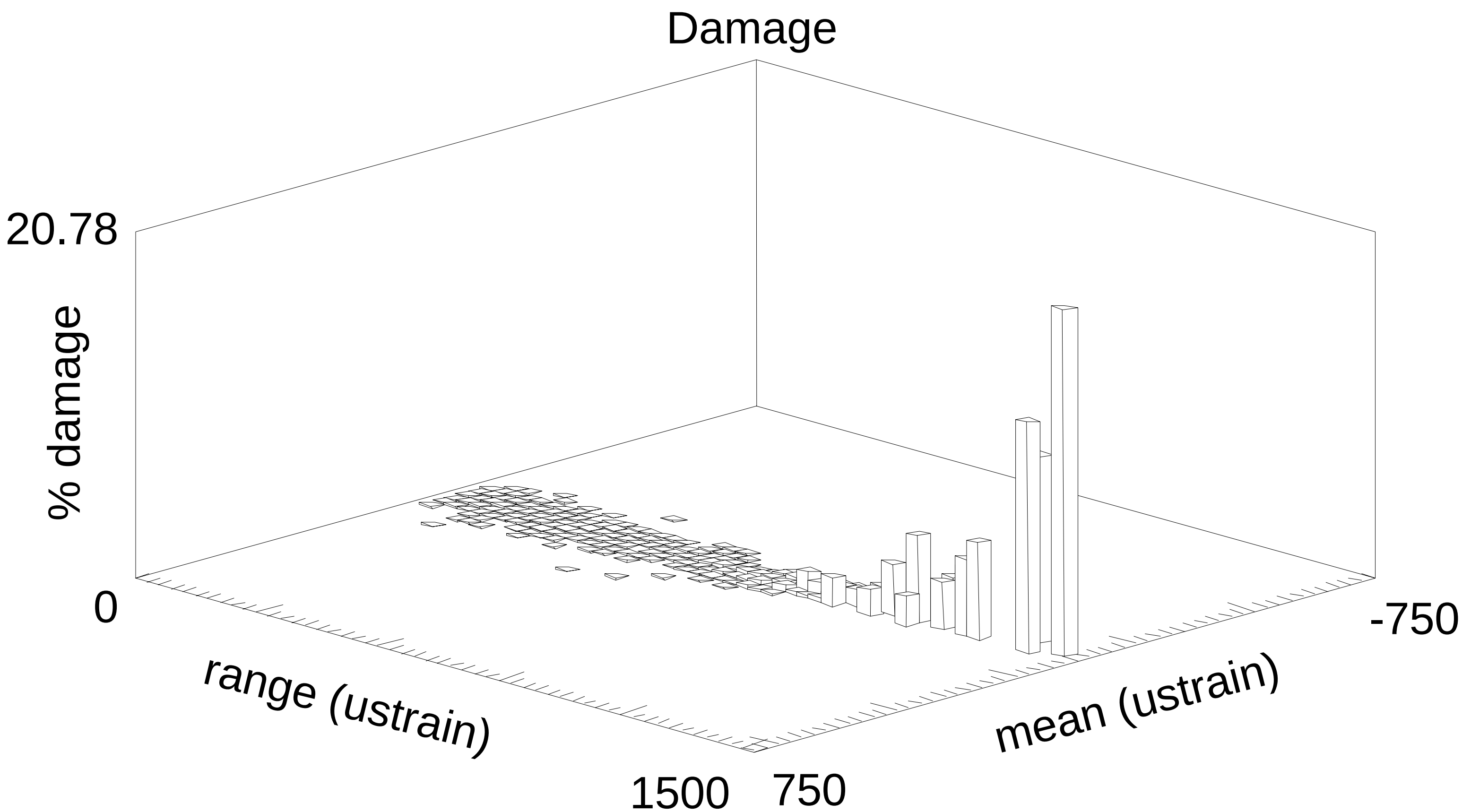


# Slope = 5



# Slope = 10


Damage
20.78
% damage
0
range (ustrain)
1500
750
mean (ustrain)
-750

# Frequency Based Fatigue